

# 8.5 Power Series

Def:  $\sum_{n=0}^{\infty} C_n(x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + \dots$   
is called a Power series centered at  $a$ .

Technique: To find the radius of convergence & the interval of convergence

- ① Use the Ratio Test to find  $R = \text{radius of convergence}$
- ② Check the convergence of the series at endpoints  $x = a - R, x = a + R$
- ③ If the series converges at an endpoint include it in your interval of convergence

Examples from Calculus: Early Transcendental Functions By Larson, Hostetler, Edwards

→ Find the radius & interval of convergence

**EX 1**  $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$

Ratio Test  $\lim_{n \rightarrow \infty} \left| \frac{\left(\frac{x}{2}\right)^{n+1}}{\left(\frac{x}{2}\right)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{2} \right| = \left| \frac{x}{2} \right| < 1$

↑ converges when  $< 1$

→  $|x| < 2$  \* Once we get it into the form  $|x-a| < R$ ,  $R = \text{Radius of convergence}$  \*

$R = 2$

P2

End points  $X = a - R \rightarrow X = 0 - 2 = -2$   
 $X = a + R \rightarrow X = 0 + 2 = 2$

$X = -2$  \* Recall: Ratio test  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \begin{cases} L < 1 \text{ converges} \\ L > 1 \text{ diverges} \\ 1 \text{ Test fails} \end{cases}$

The endpoints happen when  $|x-a| = R$ , Ratio test fails

Do NOT use the Ratio test on the endpoints — it won't work, \*

$X = -2: \sum_{n=0}^{\infty} \left(\frac{-2}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n$

Divergence test  $\lim_{n \rightarrow \infty} (-1)^n < -1 \neq 0$

diverges

$X = 2: \sum_{n=0}^{\infty} \left(\frac{2}{2}\right)^n = \sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1$  Divergence test

$\lim_{n \rightarrow \infty} 1 = 1 \neq 0$  diverges

Interval of Convergence:  $(-2, 2)$   
 diverges at endpoints so they aren't included.

Ex 2  $\sum_{n=0}^{\infty} (2n)! \left(\frac{x}{2}\right)^n$

Ratio Test  $\lim_{n \rightarrow \infty} \left| \frac{(2(n+1))! \left(\frac{x}{2}\right)^{n+1}}{(2n)! \left(\frac{x}{2}\right)^n} \right|$

$= \lim_{n \rightarrow \infty} \left| \frac{(2n+2)! \left(\frac{x}{2}\right)^{n+1}}{(2n)! \left(\frac{x}{2}\right)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)(2n+1)(2n)! \left(\frac{x}{2}\right)^{n+1}}{(2n)! \left(\frac{x}{2}\right)^n} \right|$

$$= \lim_{n \rightarrow \infty} \left| (2n+2)(2n+1) \left(\frac{x}{2}\right) \right|$$

\* To converge we need this limit to be less than 1. If  $x = -\frac{1}{100}$ , we'll have

$$\lim_{n \rightarrow \infty} \left| (2n+2)(2n+1) \left(\frac{-1}{200}\right) \right| = \infty.$$

In fact almost every value of  $x$  will let this go to  $\infty$ , except 1. When  $x=0$

$$\lim_{n \rightarrow \infty} \left| (2n+2)(2n+1) \left(\frac{0}{2}\right) \right| = \lim_{n \rightarrow \infty} \left| 0 \right| = 0 < 1$$

\* The interval of convergence is all values of  $x$  so that the series converges.

In this case I of C:  $\{0\}$   $R=0$

we can't move away from  $x=0$  without the series diverging

$$\boxed{\text{EX 3}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x+2)^n}{\sqrt{n} 4^n}$$

$$\text{Ratio Test } \lim_{n \rightarrow \infty} \left| \frac{(x+2)^{n+1}}{4^{n+1} \sqrt{n+1}} \cdot \frac{4^n \sqrt{n}}{(x+2)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x+2) \sqrt{n}}{4 \sqrt{n+1}} \right| = \left| \frac{x+2}{4} \right| < 1$$

$$|x+2| < 4 \quad R=4 \quad a=-2$$

$$\text{End points: } x = a+R \quad x = a-R$$

$$x = 2 \quad x = -6$$

$$x = -6: \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (-6+2)^n}{\sqrt{n} 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (-4)^n}{\sqrt{n} 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (-1)^n}{\sqrt{n}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{-1}{\sqrt{n}}$$

\* Note: This only looks like its Alternating - it is always negative. Check the 1<sup>st</sup> couple of terms

P4

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

p-series

$p = 1/2 < 1$  diverges

$$x = 2: \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2+2)^n}{\sqrt{n} 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cancel{4^n}}{\sqrt{n} \cancel{4^n}}$$

Alternating Series Test

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0 \quad \checkmark$$

$$C_1 = \frac{1}{\sqrt{1}} = 1 \quad C_2 = \frac{1}{\sqrt{2}} \quad C_3 = \frac{1}{\sqrt{3}}$$

$$1 > 1/\sqrt{2} > 1/\sqrt{3} \quad \checkmark$$

converges

Interval of convergence

$$(-6, 2]$$

converges at  $x = -6$   
converges at  $x = 2$

**Ex 4** a)  $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (x-1)^{n+1}}{n+1}$

Ratio test  $\lim_{n \rightarrow \infty} \left| \frac{(x-1)^{n+2}}{n+2} \cdot \frac{n+1}{(x-1)^{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-1)(n+1)}{(n+2)} \right|$

$$= |x-1| < 1$$

$\uparrow$   
 $a=1$   $R=1$

$$|x-a| < R$$

P5

Endpoints  $x = a - R$   
 $x = 1 - 1 = 0$

$x = a + R$   
 $x = 1 + 1 = 2$

$$x=0: \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{2n+2}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1}$$

\*  $n=0 \quad (-1)^{0+2} = 1$

$n=1 \quad (-1)^{2+2} = 1$

$n=2 \quad (-1)^{4+2} = 1$

etc not alternating

$$\sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \dots = \sum_{n=1}^{\infty} \frac{1}{n}$$

Harmonic series diverges

\* If you hadn't realized that this was the Harmonic series, Limit comparison would have also worked.

$$x=2: \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

$$= -1 + \frac{1}{2} - \frac{1}{3} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Alternating Harmonic series converges

\* Alternating series test would also work.

Interval of convergence  $(0, 2]$

b) Find the derivative of the series from part a) & find its radius & interval of convergence

$$f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (n+1) (x-1)^n}{n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} (x-1)^n$$

$R=1$  (Radius doesn't change when we take derivatives / integrate)

End points  $x=0, x=2$

$$x=0: \sum_{n=0}^{\infty} (-1)^{n+1} (-1)^n = \sum_{n=0}^{\infty} (-1)^{2n+1} = \sum_{n=0}^{\infty} -1$$

Divergence test  $\lim_{n \rightarrow \infty} -1 = -1 \neq 0$   
diverges

$$x=2: \sum_{n=0}^{\infty} (-1)^{n+1} (2-1)^n = \sum_{n=0}^{\infty} (-1)^{n+1}$$

Divergence test  $\lim_{n \rightarrow \infty} (-1)^{n+1} \begin{matrix} -1 \\ / \\ 1 \end{matrix} \neq 0$   
diverges

Interval of convergence

$(0, 2)$

Note: I of C doesn't necessarily stay the same.