Directional Derivatives

If $f$ is a differentiable function of $x$ and $y$, then $f$ has a directional derivative in the direction of any unit vector $\vec{u} = \langle a, b \rangle$ and

$$Duf(x, y) = f_x(x, y) a + f_y(x, y) b = \nabla f(x, y) \cdot \vec{u}$$

or if $f$ is a function of $x, y, z$

$$Duf(x, y, z) = f_x(x, y, z) a + f_y(x, y, z) b + f_z(x, y, z) c = \nabla f(x, y, z) \cdot \vec{u}$$

Ex p799 Find the directional derivative of the function at the given point in the direction of the vector $\vec{v}$.

11. $f(x, y) = 1 + 2x\sqrt{y}$ \hspace{1cm} (3, 4) \hspace{1cm} $\vec{v} = \langle 4, -3 \rangle$

First we calculate the gradient and plug in our point:

$$\nabla f = \langle f_x, f_y \rangle = \langle 2\sqrt{y}, 2x\cdot \frac{1}{2}y^{-\frac{1}{2}} \rangle$$

$$\nabla f(3, 4) = \langle 2\sqrt{4}, 3(4)^{-\frac{1}{2}} \rangle = \langle 4, \frac{3}{2} \rangle$$

Next we need to find a unit vector in the same direction as $\vec{v}$. So $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 4, -3 \rangle}{\sqrt{4^2 + (-3)^2}} = \frac{\langle 4, -3 \rangle}{5}$. 

\[ \vec{u} = \frac{\langle 4, -3 \rangle}{\sqrt{4^2 + (-3)^2}} = \frac{\langle 4, -3 \rangle}{5} = \langle \frac{4}{5}, -\frac{3}{5} \rangle \]

\[ D_{uf}(3, 4) = \nabla f(3, 4) \cdot \vec{u} = \langle 4, \frac{3}{2} \rangle \cdot \langle \frac{4}{5}, -\frac{3}{5} \rangle \]

\[ = \frac{16}{5} - \frac{9}{10} = \frac{32 - 9}{10} = \frac{23}{10} \]

15. \( g(x, y, z) = (x + 2y + 3z)^{3/2} \)

\( (1, 1, 2) \) \( \vec{v} = 2\hat{i} - \hat{k} \)

\( \text{First find gradient of } g = \nabla g = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \)

\( \nabla g = \left\langle \frac{3}{2} (x+2y+3z)^{1/2} \cdot 1, \frac{3}{2} (x+2y+3z)^{1/2} \cdot 2, \frac{3}{2} (x+2y+3z)^{1/2} \cdot 3 \right\rangle \)

\( \nabla g(1, 1, 2) = \left\langle \frac{3}{2} (1+2+6)^{1/2}, 3(1+2+6)^{1/2}, \frac{9}{4} (1+2+6)^{1/2} \right\rangle \)

\[ = \left\langle \frac{9}{2}, 9, \frac{27}{2} \right\rangle \]

\( \text{Make } \vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 0, 2, -1 \rangle}{\sqrt{0^2 + 2^2 + (-1)^2}} = \frac{\langle 0, 2, -1 \rangle}{\sqrt{5}} = \langle 0, \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \rangle \)

\( \text{Then } D_{ug}(1, 1, 2) = \nabla g(1, 1, 2) \cdot \vec{u} = \left\langle \frac{9}{2}, 9, \frac{27}{2} \right\rangle \cdot \langle 0, \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \rangle \)

\[ = \frac{9}{2} (0) + 9 \left( \frac{2}{\sqrt{5}} \right) + \frac{27}{2} \left( -\frac{1}{\sqrt{5}} \right) \]

\[ = \frac{18}{\sqrt{5}} - 13.5 = \frac{4.5}{\sqrt{5}} \]
Maximizing the Directional Derivative

Thm: Suppose \( f \) is a differentiable function of 2 or 3 variables. The maximum value of the directional derivative \( D_u f(x,y) \) (or \( D_u f(x,y,z) \)) is \( |\nabla f(x,y)| \) (or \( |\nabla f(x,y,z)| \)) and it occurs when \( u \) has the same direction as \( \nabla f \).

What this means:

- The direction of the maximum rate of change is \( \nabla f \) evaluated at our point.
- The maximum rate of change is \( |\nabla f| \) evaluated at our point.

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47. Find the maximum rate of change of \( f(x,y) = x^2y + \sqrt{y} \) at the point \((2,1)\). In which direction does it occur?
First we find the gradient 

\[ \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = \left( 2xy, x^2 + \frac{1}{2}y^2 \right) \]

\[ \nabla f(2,1) = \left( 2(2)(1), 2^2 + \frac{1}{2}(1)^2 \right) = \left( 4, \, 4.5 \right) \]

The direction of the maximum rate of change is \( \nabla f(2,1) = \left< 4, \, 4.5 \right> \)

* If they had asked for this as a unit vector (or as a vector of magnitude 1), our answer would have been 

\[ \frac{\nabla f(2,1)}{|\nabla f(2,1)|} = \frac{\left< 4, \, 4.5 \right>}{\sqrt{4^2 + (4.5)^2}} = \frac{\left< 4, \, 4.5 \right>}{\sqrt{16 + \frac{81}{4}}} \]

\[ = \frac{\left< 4, \, 4.5 \right>}{\sqrt{\frac{64}{4} + \frac{81}{4}}} = \frac{\left< 4, \, 4.5 \right>}{\sqrt{\frac{145}{4}}} = \frac{2\left< 4, \, 4.5 \right>}{\sqrt{145}} \]

The maximum rate of change of 

\( f \) at \( (2,1) \) is 

\[ |\nabla f(2,1)| = \sqrt{\frac{145}{4}} = \frac{\sqrt{145}}{2} \]
Ex From Calculus 6e by Edwards & Penney

P916 Find the maximum directional derivative of \( f \) at \( P \) and the direction in which it occurs.

21. \( f(x, y) = 2x^2 + 3xy + 4y^2 \) \( P(1, 1) \)

→ Find the gradient \( \nabla f = \langle \frac{df}{dx}, \frac{df}{dy} \rangle \)

\[ = \langle 4x + 3y, 3x + 8y \rangle \]

→ Find the gradient at \( (1, 1) \)

\( \nabla f(1, 1) = \langle 4+3, 3+8 \rangle \)

\[ = \langle 7, 11 \rangle \]

The direction of the maximum directional derivative is \( \nabla f(1, 1) = \langle 7, 11 \rangle \)

The maximum directional derivative is

\[ |\nabla f(1, 1)| = |\langle 7, 11 \rangle| = \sqrt{49 + 121} = \sqrt{170} \]

Alternatively, we could have \( \vec{u} = \frac{\nabla f(1, 1)}{|\nabla f(1, 1)|} = \frac{\langle 7, 11 \rangle}{\sqrt{170}} \)

\[ D_{\vec{u}} f(1, 1) = \nabla f(1, 1) \cdot \vec{u} = \langle 7, 11 \rangle \cdot \frac{\langle 7, 11 \rangle}{\sqrt{170}} \]

\[ = \frac{49}{\sqrt{170}} + \frac{121}{\sqrt{170}} = \frac{170}{\sqrt{170}} \cdot 1 = \frac{170}{\sqrt{170}} \cdot \frac{\sqrt{170}}{\sqrt{170}} = \sqrt{170} \]
Ex. Find the maximum rate of change of the function at the given point. In which direction does it occur?

\[ f(x, y, z) = e^{x-y-z} \quad P(5, 2, 3) \]

\[ \rightarrow \text{Find } \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \]

\[ = \left\langle e^{x-y-z}, \quad e^{x-y-z}, \quad e^{x-y-z} \right\rangle \]

\[ = \left\langle e^{x-y-z}, \quad -e^{x-y-z}, \quad -e^{x-y-z} \right\rangle \]

\[ \rightarrow \quad \nabla f(5, 2, 3) = \left\langle e^{5-2-3}, \quad -e^{5-2-3}, \quad -e^{5-2-3} \right\rangle \]

\[ = \left\langle e^0, \quad -e^0, \quad -e^0 \right\rangle \]

\[ = \left\langle 1, \quad -1, \quad -1 \right\rangle \]

The direction of the maximum rate of change \( \nabla f(5, 2, 3) = \left\langle 1, -1, -1 \right\rangle \)

\[ \rightarrow \quad \text{The maximum rate of change is} \]

\[ |\nabla f(5, 2, 3)| = \sqrt{1^2 + (-1)^2 + (-1)^2} = \sqrt{3} \]