Limit Comparison Test

Suppose $a_n > 0$, $b_n > 0$ and $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = L$ where L is finite and $L > 0$, then the 2 series $\sum a_n$ and $\sum b_n$ both converge or both diverge

* I will be re-doing problems from the Comparison Test WS with the Limit Comparison Test when applicable

Ex 1 $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

* n^{0-2} ← degree of top, degree of bottom $= n^{-2} = \frac{1}{n^2}$

$$\lim_{n\to\infty} \left(\frac{\frac{1}{n^2+1}}{\frac{1}{n^2}}\right) = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 > 0$$

Both converge or diverge by the Limit Comparison Test. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ p-series, $p = 2 > 1$ converges

so $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges

$\boxed{\text{Ex 2}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$ $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{3n^2+2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{3n^2+2} = \frac{1}{3} > 0$

since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ p-series, $p=2>1$ converges

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$ Converges by the Limit Comparison test

$\boxed{\text{Ex 3}}$ $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n-2}$ $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{n-2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n-2} = 1 > 0$

Since $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$ Harmonic series diverges

$\displaystyle\sum_{n=3}^{\infty} \frac{1}{n-2}$ diverges by the Limit Comparison Test

* This is the minimal work needed to get full credit on this problem using this test *

$\boxed{\text{Ex 4}}$ $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n+1}$ $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{3^n+1}}{\frac{1}{3^n}} = \lim_{n\to\infty} \frac{3^n}{3^n+1} = 1 > 0$

$\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n}$ Geometric series $r=\frac{1}{3}$ $\left|\frac{1}{3}\right|<1$ converges

So $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n+1}$ converges by the Limit Comparison test

$\boxed{\text{Ex 5}}$ $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3^n+5}$

$$\lim_{n\to\infty} \frac{\frac{2^n}{3^n+5}}{\frac{2^n}{3^n}} = \lim_{n\to\infty} \frac{3^n 2^n}{(3^n+5)2^n} = 1 > 0$$

$\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$ Geometric series $r = 2/3$

$|2/3| < 1$ converges

So $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3^n+5}$ converges by the Limit Comparison test

$\boxed{\text{Ex 6}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

degree of top $= 0$
degree of bottom $= 3/2$ $\left(\sqrt{n^3+1} = (n^3+1)^{1/2}\right)$

$$\lim_{n\to\infty} \left(\frac{\frac{1}{\sqrt{n^3+1}}}{\frac{1}{n^{3/2}}} \right) = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3+1}} = 1 > 0$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ p-series $p = 3/2 > 1$ converges

So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ converges as well by the LCT

$\boxed{\text{Ex 7}}$ $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-1}}$

$$\lim_{n\to\infty} \left(\frac{\frac{1}{\sqrt{n^3-1}}}{\frac{1}{n^{3/2}}} \right) = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3-1}} = 1 > 0$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ p-series $p = 3/2 > 1$ converges

So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3-1}}$ converges as well by the limit comparison test

$\boxed{\text{EX 8}}$ $\sum_{n=1}^{\infty} \frac{4^n}{3^n-1}$ $\lim_{n\to\infty} \frac{\frac{4^n}{3^n-1}}{\frac{4^n}{3^n}} = \lim_{n\to\infty} \frac{4^n 3^n}{(3^n-1)4^n} = 1 > 0$

$\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$ Geometric series $r = \frac{4}{3}$ $|4/3| > 1$

diverges

so $\sum_{n=1}^{\infty} \frac{4^n}{3^n-1}$ diverges as well by the Limit Comparison Test

$\boxed{\text{Ex 9}}$ $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$ $\lim_{n\to\infty} \frac{\frac{1}{3^n+2}}{\frac{1}{3^n}} = \lim_{n\to\infty} \frac{3^n}{3^n+2} = 1 > 0$

$\sum_{n=1}^{\infty} \frac{1}{3^n}$ Geometric series with $r = 1/3$

$|1/3| < 1$ converges

so $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$ converges by Limit Comparison test