Integration By Parts

The goal of integration by parts is to undo the product rule:

$$\frac{d}{dx}[fg] = f'g + fg'$$

$$\int \frac{d}{dx}[fg]\,dx = \int f'g\,dx + \int fg'\,dx$$

$$fg = \int f'g\,dx + \int fg'\,dv$$

OR

$$\int fg'\,dx = fg - \int f'g\,dx$$

But we usually write the formula as

$$\boxed{\int u\,dV = uv - \int v\,du}$$

where $u = f$ so $du = f'dx$

$v = g$ $\quad dv = g'dx$

To find u, use LIATE

logs

Inverse trig

algebraic (polynomials & things like polynomials ex x^{-2}, $\sqrt[3]{x}$, etc)

trig

exponentials

By picking u in this order we make dV easy (relatively easy) to integrate.

Technique: $\int u\,dV = uv - \int v\,du$

① Pick u by LIATE

② dV is everything left over

③ Plug into the formula

✳ There are lots of times when integration by parts works, but <u>mostly</u> we do it for problems of the following types:

$\int x^n \cos x\,dx$

$\int x^n \sin x\,dx$

$\int x^n \ln x\,dx$ (But <u>not</u> $\int \frac{\ln x}{x}\,dx$ this is a u-sub w/ $u=\ln x$)

$\int x^n e^x\,dx$

$\left.\begin{array}{l}\int e^x \sin x\,dx \\ \int e^x \cos x\,dx\end{array}\right\}$ these are a hassle but strangely enjoyable. You should have done one of them in Calc 1. Unfortunately they are too hard for a test. :(

Ex 1 $\int (x+2)\cos x \, dx$

* we can tell that it is integration by parts since it is of the type $\int x^n \cos x \, dx$

→ Pick u by LIATE

no logs

no inverse trig

yes

differentiate

$u = x+2$ $\quad$ $v = \sin x$ $\quad$ take antiderivative of dV to find V

$du = dx$ $\quad$ $dV = \underbrace{\cos x \, dx}$ ← everything left after we find u

$$\int u \, dV = uv - \int v \, du$$

$$\int (x+2)\cos x \, dx = (x+2)\sin x - \int \sin x \, dx$$

$$= \boxed{(x+2)\sin x + \cos x + C}$$

Ex 2 $\int x \sin(3x) \, dx$

* Of the type $\int x^n \sin x \, dx$

→ Pick u by LIATE

take derivative

$u = x$ $\quad$ $V = -\frac{1}{3}\cos(3x)$ $\quad$ take the antiderivative

$du = dx$ $\quad$ $dV = \sin(3x)\,dx$

$$\int u \, dV = uv - \int v \, du$$

$$\int x \sin(3x)\,dx = \frac{-x}{3}\cos(3x) - \int -\frac{1}{3}\cos(3x)\,dx$$

$$= \frac{-x}{3}\cos(3x) + \int \frac{1}{3}\cos 3x \, dx = \boxed{\frac{-x}{3}\cos 3x + \frac{1}{9}\sin 3x + C}$$

$\boxed{\text{Ex } 3}$ $\int (x^3+x^2)\ln x \, dx$

→ Pick u by LIATE

so $u = \ln x$ $\qquad v = \left(\frac{1}{4}x^4 + \frac{1}{3}x^3\right)$ take antiderivative

take derivative $\quad du = \frac{1}{x}dx \qquad dv = (x^3+x^2)\,dx$

(everything we left after find u

$$\int (x^3+x^2)\ln x\,dx = uv - \int v\,du$$

$$= (\ln x)\left(\frac{1}{4}x^4 + \frac{1}{3}x^3\right) - \int \left(\frac{1}{4}x^4 + \frac{1}{3}x^3\right)\frac{1}{x}\,dx$$

$$= \ln x\left(\frac{1}{4}x^4 + \frac{1}{3}x^3\right) - \int \frac{1}{4}x^3 + \frac{1}{3}x^2\,dx$$

$$= \ln x\left(\frac{1}{4}x^4 + \frac{1}{3}x^3\right) - \left(\frac{1}{16}x^4 + \frac{1}{9}x^9\right) + C$$

$$= \boxed{\ln x\left(\frac{1}{4}x^4 + \frac{1}{3}x^3\right) - \frac{1}{16}x^4 - \frac{1}{9}x^9 + C}$$

* $\int x^n \ln x\,dx$ is one of the easiest integration by parts problems, no matter how weird we make x^n it is still easy to do.

$\boxed{\text{Ex 4}}$ $\int_1^4 \sqrt{x} \ln x \, dx$

Definite integrals (integrals w/ endpoints) do make integration by parts a little harder. Basically all we need to do is evaluate after we integrate.

→ Pick u by LIATE

$u = \ln x \qquad v = \frac{2}{3}x^{3/2}$

$du = \frac{1}{x}dx \qquad dv = \sqrt{x}\,dx = x^{1/2}dx$

$$\int_1^4 \sqrt{x}\ln x\,dx = \underbrace{\frac{2}{3}x^{3/2}}_{v}\underbrace{\ln x}_{u}\Big|_1^4 - \int_1^4 \underbrace{\frac{2}{3}x^{3/2}}_{v}\underbrace{\frac{1}{x}dx}_{du}$$

$$\frac{2}{3}(4)^{3/2}\ln 4 - \cancel{\frac{2}{3}1^{3/2}\ln 1} - \int_1^4 \frac{2}{3}x^{1/2}dx$$

$$\frac{2}{3}(8)\ln 4 - \frac{4}{9}x^{3/2}\Big|_1^4$$

$$\frac{16}{3}\ln 4 - \left(\frac{4}{9}\cdot 4^{3/2} - \frac{4}{9}\cdot 1^{3/2}\right)$$

$$\boxed{\frac{16}{3}\ln 4 - \frac{4}{9}(8) + \frac{4}{9}}$$

$\boxed{\text{Ex 5}}$

$$\int_0^1 xe^{-2x}\,dx$$

→ Choose u by LIATE

$u = x \qquad v = -\frac{1}{2}e^{-2x}$

$du = dx \qquad dv = e^{-2x}\,dx$

$$\int_0^1 xe^{-2x}\,dx = -\frac{x}{2}e^{-2x}\Big|_0^1 - \int_0^1 -\frac{1}{2}e^{-2x}\,dx$$

$$uv\Big|_0^1 - \int_0^1 v\,du$$

$$= -\frac{1}{2}e^{-2} - \cancel{\left(-\frac{0}{2}e^{-2\cdot 0}\right)} + \int_0^1 \frac{1}{2}e^{-2x}\,dv$$

$$= -\frac{1}{2e}^{-2} - \frac{1}{4}e^{-2x}\Big|_0^1$$

$$-\frac{1}{2e}^{-2} - \frac{1}{4}e^{-2} + \frac{1}{4}e^{-2\cdot 0}$$

$$-\frac{1}{2e}^{-2} - \frac{1}{4}e^{-2} + \frac{1}{4}$$

$$\boxed{-\frac{3}{4}e^{-2} + \frac{1}{4}}$$

$\boxed{\text{Ex 6}}$ $\int (x^2+2)e^x\,dx$

* This would be an ideal candidate for Tabular Integration by parts

Short cut that works if u = polynomial

→ Choose u by LIATE

$u = x^2+2 \qquad v = e^x$

$du = 2x\,dx \qquad dv = e^x\,dx$

$$\int (x^2+2)e^x\,dx = uv - \int v\,du$$
$$= (x^2+2)e^x - \int e^x(2x)\,dx$$

to integrate this we do integration by parts again!

$$\int e^x(2x)\,dx$$

$u = 2x \qquad v = e^x$

$du = 2dx \qquad dv = e^x\,dx$

(you could also do $u = x$ & $dv = 2e^x$ & get the same result)

$$\int e^x(2x)\,dx = uv - \int v\,du$$
$$= 2xe^x - \int e^x 2\,dx$$
$$= 2xe^x - 2e^x + C$$

Plug back in:

$$\int (x^2+2)e^x\,dx = (x^2+2)e^x - \int e^x 2x\,dx$$
$$= (x^2+2)e^x - [2xe^x - 2e^x + C]$$
$$= \boxed{(x^2+2)e^x - 2xe^x + 2e^x + C}$$

C is just a constant so $-C$ would still just be a constant