Integral Test

Requirements: $\sum_{n=1}^{\infty} a_n$ $a_n = f(n)$ where f is continuous, positive and decreasing $[1, \infty)$

→ If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent

→ If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent

Examples from *Calculus: The Language of Change* by Cohen & Henle

Use the Integral Test to determine whether the series converges or diverges

EX 1 $\sum_{n=1}^{\infty} \frac{1}{n^3}$

* This is a p-series with $p = 3 > 1$ so we know this converges; however, they want us to show this with the Integral test. Had they not specified integral test, we should have just used p-series.

We look at $\int_1^\infty \frac{1}{x^3}dx$ Basically we just replace the n's with x's

$$\int_1^\infty \frac{1}{x^3}dx = \lim_{n\to\infty} \int_1^n x^{-3}dx = \lim_{n\to\infty} \frac{x^{-3+1}}{-3+1}\Big|_1^n$$

$$= \lim_{n\to\infty} \frac{x^{-2}}{-2}\Big|_1^n = \lim_{n\to\infty} -\frac{1}{2}\left(n^{-2}-1^{-2}\right)$$

$$= \lim_{n\to\infty} -\frac{1}{2}\left(\frac{1}{n^2}-1\right) = -\frac{1}{2}(0-1)$$

$$= \boxed{\frac{1}{2}} \text{ converges}$$

Since the improper integral converges, so does $\sum_{n=1}^\infty \frac{1}{n^3}$

Ex 2 $\sum_{n=2}^\infty \frac{1}{n\ln(n)}$

* Besides that they told us to do Integral test, we can see that a_n is positive & decreasing but most importantly we can see that we can integrate it

$$\int_2^\infty \frac{1}{x\ln x}dx$$

start the where series starts

To integrate this improper Integral we need a u-substitution. To make our lives easier, 1st we'll do the u-sub & then we'll use the limit.

$u = \ln x \qquad u(2) = \ln 2$

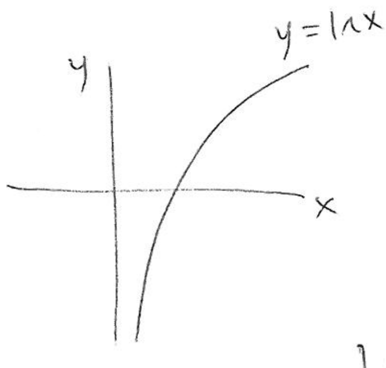
$du = \frac{1}{x}dx \qquad u(\text{"}\infty\text{"}) = \ln \text{"}\infty\text{"} = \infty$

we can't plug in ∞ since it is the concept of increasing without bound, but we can see what happens if x increases without bound.

$$\int_2^\infty \frac{1}{x \ln x}\,dx = \int_{\ln 2}^{\infty} \frac{1}{u}\,du = \lim_{n\to\infty} \int_{\ln 2}^{n} \frac{1}{u}\,du$$

$$= \lim_{n\to\infty} \ln|u| \Big|_{\ln 2}^{n}$$

$$= \lim_{n\to\infty} \ln(n) - \underbrace{\ln(\ln 2)}$$



looks weird, but it is just a number

We know $\lim\limits_{n\to\infty} \ln(n) \to \infty$

$= \infty$ diverges

So $\sum\limits_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges as well

Ex 3 $\sum_{n=1}^{\infty} ne^{-n^2}$

$\int_1^{\infty} xe^{-x^2}dx$ Another u-sub problem:

$$u=-x^2 \qquad u(1)=-(1)^2=-1$$
$$du=-2x\,dx \qquad u(\text{"}\infty\text{"})=-(\text{"}\infty\text{"})^2=-\infty$$
$$-\tfrac{1}{2}du=x\,dx$$

$$\int_1^{\infty} xe^{-x^2}dx=\int_{-1}^{-\infty}-\tfrac{1}{2}e^u\,du=\int_{-\infty}^{-1}\tfrac{1}{2}e^u\,du$$

$$\left(\int_a^b f(x)dx=-\int_b^a f(x)dx\right)$$

$$=\lim_{n\to-\infty}\int_n^{-1}\tfrac{1}{2}e^u\,du=\lim_{n\to-\infty}\tfrac{1}{2}e^u\Big|_n^{-1}$$

$$\lim_{n\to-\infty}\tfrac{1}{2}e^{-1}-\tfrac{1}{2}e^n=\boxed{\tfrac{1}{2}e^{-1}} \quad \text{converges}$$

$$\left(\lim_{n\to-\infty}e^n=\lim_{n\to\infty}\frac{1}{e^n}=0\right)$$

So $\sum_{n=1}^{\infty} ne^{-n^2}$ converges as well

* Note: $\sum_{n=1}^{\infty} ne^{-n^2}$ does <u>NOT</u> converge to $\frac{1}{2}e^{-1}$. we only know it converges