Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. If $\sum b_n$ is convergent AND $a_n \leq b_n$ for all $n$, then $\sum a_n$ is also convergent.
2. If $\sum b_n$ is divergent AND $a_n \geq b_n$ for all $n$, then $\sum a_n$ is also divergent.

Examples from Calculus: Early Transcendental Functions By Larson, Hostetler, Edwards

**Ex1**
\[\sum_{n=1}^{\infty} \frac{1}{n^{2+1}}\] * We want to compare this to a p-series or a geometric series.

If we remove something positive from the denominator the fraction gets bigger,

\[
\left(\frac{1}{2+1} \leq \frac{1}{2}\right) *
\]

\[\sum_{n=1}^{\infty} \frac{1}{n^{2+1}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{2}}\]

p-series $p=2>1$ converges

So $\sum_{n=1}^{\infty} \frac{1}{n^{2+1}}$ converges by the Comparison Test

Note:
\[\sum_{n=1}^{\infty} \frac{1}{n^{2+1}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{1}}\]

but $\sum_{n=1}^{\infty} \frac{1}{n^{1}}$ diverges so this isn't helpful. Being less than something that diverges doesn't mean that the series converges or diverges.
Ex 2 \[ \sum_{n=1}^{\infty} \frac{1}{3n^2+2} \leq \sum_{n=1}^{\infty} \frac{1}{3n^2} \quad \text{p-series } p=2>1 \]

So \[ \sum_{n=1}^{\infty} \frac{1}{3n^2+2} \] converges by comparison test.

Ex 3

\[ \sum_{n=3}^{\infty} \frac{1}{n-2} \]

* If we remove something negative from the denominator, the fraction gets smaller \( \left( \frac{1}{2-1} = 1, \quad \frac{1}{2-1} \geq \frac{1}{2} \right) \). If you get confused, try using numbers to figure out the inequality.*

\[ \sum_{n=3}^{\infty} \frac{1}{n-2} \geq \sum_{n=2}^{\infty} \frac{1}{n} \quad \text{Harmonic series diverges} \]

So \[ \sum_{n=3}^{\infty} \frac{1}{n-2} \] diverges by the comparison test.

Ex 4

\[ \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} \leq \sum_{n=0}^{\infty} \frac{1}{3^n} = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots \quad \text{Geometric series} \]

\[ r = \frac{1}{3}, \quad \left| \frac{1}{3} \right| < 1 \]

So \[ \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} \] converges by the comparison test.

Ex 5

\[ \sum_{n=0}^{\infty} \frac{2^n}{3^{n+5}} \leq \sum_{n=0}^{\infty} \frac{2^n}{3^n} = 1 + \frac{2}{3} + \left( \frac{2}{3} \right)^2 + \cdots \quad \text{Geometric series} \]

\[ r = \frac{2}{3}, \quad \left| \frac{2}{3} \right| < 1 \]

So \[ \sum_{n=0}^{\infty} \frac{2^n}{3^{n+5}} \] converges by the comparison test.
\[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}} \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \]

p-series \( p = \frac{3}{2} > 1 \) converges.

So converges by the comparison test.

\[ \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-1}} \]

* I can't remove the one because \( \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-1}} \geq \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3}} \) Being bigger than something that converges doesn't tell us anything.*

\[ \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-1}} \leq \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-\frac{1}{2}n^3}} = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3}} = \sum_{n=2}^{\infty} \frac{1}{n^{\frac{1}{2}}n^{\frac{3}{2}}} \]

p-series \( p = \frac{3}{2} > 1 \) converges.

So \( \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-1}} \) converges by the comparison test.

* I wanted to get to \( \frac{1}{\sqrt{n^3}} \) so I knew that I had to subtract something bigger than 1 that was a multiple of \( \frac{1}{\sqrt{n^3-\frac{1}{2}n^3}} \) obviously doesn't work so

1 tried \( \frac{1}{2}n^3 \)

\( n=2 \quad \frac{1}{2}n^3 = \frac{8}{4} = 2 > 1 \)

\( n=3 \quad \frac{1}{2}n^3 = \frac{27}{6} > 1 \)
\[ \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} \frac{4^n}{3^n} = \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^3 + \ldots \]

Geometric series \( r = \frac{4}{3} \quad \left| \frac{4}{3} \right| > 1 \)

So \( \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} \) diverges by Comparison Test.

\[ \sum_{n=1}^{\infty} \frac{1}{3^{n+2}} = \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \ldots \]

Geometric series \( r = \frac{1}{3} \quad \left| \frac{1}{3} \right| < 1 \)

Converges by Comparison Test.