MA 242 Test 4 Version 1

1. (5 points) Find a parametric representation for the part of the ellipsoid $3x^2+5y^2+8z^2=1$ that lies to the right of the xz - plane.

2. (8 points) Let S be the surface $x^2+z^2=81$ with $-13\le y\le 24$. Find $\vec{r}(u,v)$ and list the appropriate bounds for u and v if $\vec{r}(u,v)$ is a parametrization of the surface S using the polar transformation equations $x=R\cos(v)$, $z=R\sin(v)$, $y=u$

3. (12 points) Find the area of the hyperbolic paraboloid $z=y^2-x^2$ between the cylinders $x^2+y^2=1$ and $x^2+y^2=3$

4. (5 points) Find the gradient vector field of $f(x,y,z)=x^2y+\ln(4-3z)$

5. (29 points) a) Evaluate the line integral $\int_{C_1} -y\,dx+x\,dy$ along C_1 where C_1 is the top half of the circle $x^2+y^2=4$ from (2,0) to (-2,0)

b) Evaluate the line integral $\int_{C_2} -y\,dx+x\,dy$ along C_2 where C_2 is the line segment from (-2,0) to (-8, 18)

c) Use your work from a) and b) to find the line integral of $\int_C -y\,dx+x\,dy$ where C consists of C_1 and C_2

6. (12 points) Find the mass of the wire bent into the shape of $y=\sqrt{9-x^2}$ from (0,3) to $(2,\sqrt{5})$ if the density $\rho(x,y)=x^3y$. Show all of your work.

7. (29 points) Use $\vec{\mathbf{F}}(x,y,z)=(2xy+\cos(x))\mathbf{i}+(x^2-e^{3z}\sin(y))\mathbf{j}+\left(3e^{3z}\cos(y)+5\right)\mathbf{k}$ to answer the following.

a) Show $\vec{\mathbf{F}}$ is conservative

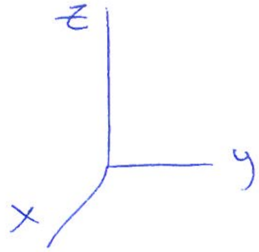
b) Find the most general potential function f of $\vec{\mathbf{F}}$

c) Find the work done by $\vec{\mathbf{F}}(x,y,z)$ on a particle that moves along $\vec{r}(t)=\langle t^3,\pi t,t(t-1)\rangle$ where $0\le t\le 1$

d) Find div $\vec{\mathbf{F}}$

C3T4V1 Solutions

1. (5pts)



right of xz $\rightarrow$ $y \geq 0$

$$y = \sqrt{\tfrac{1}{5} - \tfrac{3}{5}x^2 - \tfrac{8}{5}z^2}$$

$$\vec{r}(x,z) = \left\langle x, \sqrt{\tfrac{1}{5} - \tfrac{3}{5}x^2 - \tfrac{8}{5}z^2}, z \right\rangle$$

2. (8pts)

$$\vec{r}(u,v) = \langle 9\cos v, u, 9\sin v \rangle$$

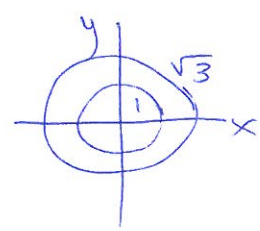
$$-13 \leq u \leq 24$$
$$0 \leq v \leq 2\pi$$

3. (12 pts)

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

$$= \sqrt{(-2x)^2 + (2y)^2 + 1}$$

$$= \sqrt{4x^2 + 4y^2 + 1}$$



$$S.A. = \iint \sqrt{4x^2+4y^2+1}\, dA$$
$$= \int_0^{2\pi} \int_1^{\sqrt{3}} \sqrt{4r^2+1}\; r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_1^{\sqrt{3}} \sqrt{4r^2+1}\; \underline{r\,dr}\, d\theta$$

$u = 4r^2+1$ $\quad u(1)=5$

$du = 8r\,dr$ $\quad u(\sqrt{3})=13$

$\frac{1}{8}du = r\,dr$

$$\int_0^{2\pi}\int_5^{13} \frac{1}{8}\sqrt{u}\, du\, d\theta$$

$$2\pi \int_5^{13} \frac{1}{8} u^{1/2} du$$

$$2\pi \; \frac{1}{8}\,\frac{2}{3}\, u^{3/2} \Big|_5^{13}$$

$$\boxed{\frac{\pi}{6}\left(13^{3/2}-5^{3/2}\right)}$$

4. (5 points)

$\nabla f = \langle f_x, f_y, f_z \rangle$

$= \left\langle 2xy, x^2, \frac{-3}{4-3z} \right\rangle$

5. (29 pts) a)

C_1: $\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle \quad 0 \le t \le \pi$

$$\int_{C_1} -2\sin t(-2\sin t) + 2\cos t \, 2\cos t \, dt$$

$$= \int_0^{\pi} 4\sin^2 t + 4\cos^2 t \, dt$$

$$= \boxed{4\pi}$$

b) C_2: $\vec{r}(t) = (1-t)\langle -2, 0 \rangle + \langle -8, 18 \rangle t$

$$= \langle -2+2t, 0 \rangle + \langle -8t, 18t \rangle$$

$$= \langle -2-6t, 18t \rangle$$

$$\int_0^1 -(18t)(-6) + (-2-6t)(18) \, dt$$

$$= \int_0^1 \cancel{18(6)t} - 18(2) - \cancel{18 \cdot 6t} \, dt$$

$$= -36t \Big|_0^1 = \boxed{-36}$$

c) $\int_C -y\,dx + x\,dy = \boxed{4\pi - 36}$

6. (12 pts) $m = \int \rho(x,y)\,ds$

$\vec{r}(t) = \langle t, \sqrt{9-t^2} \rangle \quad \vec{r}\,'(t) = \langle 1, \frac{-t}{\sqrt{9-t^2}} \rangle$

$0 \le t \le 2$

$$m = \int_0^2 t^3 \sqrt{9-t^2} \sqrt{1^2 + \frac{t^2}{9-t^2}}\, dt$$

$$= \int_0^2 t^3 \sqrt{9-t^2} \sqrt{\frac{9 \cancel{-t^2} \cancel{+t^2}}{9-t^2}}$$

$$= \int_0^2 t^3 \cancel{\sqrt{9-t^2}} \frac{3}{\cancel{\sqrt{9-t^2}}}\, dt$$

$$= \frac{3}{4} t^4 \Big|_0^2 = \boxed{12}$$

7. (29 pts)

a) $\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + \cos x & x^2 - e^{3z}\sin y & 3e^{3z}\cos y + 5 \end{vmatrix}$

$$= \langle -3e^{3z}\sin y - (-3e^{3z}\sin y), -(0-0), 2x - 2x \rangle$$

$$= \vec{0} \checkmark$$

7b

$f_x = 2xy + \cos x \qquad f_y = x^2 - e^{3z}\sin(y) \qquad f_z = 3e^{3z}\cos(y) + 5$

$f = x^2y + \sin x + g(y,z)$

$f_y = \underline{x^2} + g_y(y,z) = \underline{x^2} - e^{3z}\sin(y)$

$g(y,z) = e^{3z}\cos(y) + h(z)$

$f = x^2y + \sin x + e^{3z}\cos(y) + h(z)$

$f_z = 3e^{3z}\cos(y) + h'(z) = 3e^{3z}\cos(y) + 5$

$h(z) = 5z + k$

$$\boxed{f = x^2y + \sin x + e^{3z}\cos(y) + 5z + k}$$

7c. $W = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))$

$\vec{r}(1) = \langle 1, \pi, 0 \rangle$

$\vec{r}(0) = \langle 0, 0, 0 \rangle$

$= f(1,\pi,0) - f(0,0,0)$

$= \pi + \sin 1 + e^0\cos\pi + 0 - [0 + \cancel{\sin 0} + e^0\cos 0 + \cancel{5\cdot 0}]$

$$\boxed{\pi + \sin 1 - 1 - 1}$$

d) $\operatorname{div}\vec{F} = \nabla \cdot \vec{F}$

$$= 2y - \sin x - e^{3z}\cos y + 9e^{3z}\cos(y))$$