MA 242 Honors Test 4

1. (15 points) Evaluate $\int_C xy\,dx + (x+y)\,dy$ where C is the part of the graph $y = x^2$ from (0,0) to (2,4).

2. (30 points) a) State the Fundamental Theorem for Line Integrals

b) Assume $\vec{\mathbf{F}}$ is a conservative vector field, find f so that $\vec{\mathbf{F}} = \nabla f$

$$\vec{\mathbf{F}}(x,y,z) = (\sin(x) + \ln(y))\mathbf{i} + \left(\frac{x}{y} - e^z\right)\mathbf{j} + \left(4 - ye^z\right)\mathbf{k}$$

c) Use your answer from part b) to find the work done by the force field $\vec{\mathbf{F}}$ in moving an object from (0,1,0) to (0,1,1) along any smooth curve C.

d) Use $\vec{\mathbf{F}}(x,y,z)$ from part b) to find the div $\vec{\mathbf{F}}$

3. (15 points) Find the mass of a wire with density $\rho(x,y) = xy$ shaped like the first-quadrant portion of the circle $x^2 + y^2 = 25$

4. (12 points) Use Green's theorem $\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$ to find the work done by the force $\vec{\mathbf{F}}(x,y) = (2x + \tan x)\mathbf{i} + (x^2 + \sqrt{y})\mathbf{j}$ on a particle that is moving counterclockwise around the boundary of the triangle with vertices (0,0), (1,1), (2,0).

5. (13 points) If $\vec{\mathbf{F}}(x,y,z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is conservative with continuous second-order partial derivatives, show that curl $\vec{\mathbf{F}} = \vec{\mathbf{0}}$.

6. (15 points) Find the surface area of the paraboloid $z = 8 - x^2 - y^2$ that lies within the cylinder $x^2 + y^2 = 1$

# C3 H T4 Solutions

(15pts) 1. $\int_C xy\,dx + (x+y)\,dy$

$dx \to x'(t)\,dt \qquad dy \to y'(t)\,dt$

$y = x^2 \to \vec{r}(t) = \langle t, t^2 \rangle$, with $x(t) = t$, $y(t) = t^2$

$\int_0^2 t t^2\,dt + (t + t^2)\,2t\,dt$

$= \int_0^2 t^3 + 2t^2 + 2t^3\,dt = \int_0^2 3t^3 + 2t^2\,dt$

$= \frac{3}{4}t^4 + \frac{2}{3}t^3 \Big|_0^2 = \boxed{12 + \frac{16}{3}}$

(30pts) 2. a) $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

b)

$f_x = \sin x + \ln y \qquad f_y = \frac{x}{y} - e^z \qquad f_z = 4 - ye^z$

$f = -\cos x + x\ln y + g(y,z)$

$f_y = \frac{x}{y} + g_y(y,z) = \frac{x}{y} - e^z$

$g_y = -e^z$

$g = -ye^z + h(z)$

$f = -\cos x + x\ln y - ye^z + h(z)$

$f_z = -ye^z + h'(z) = 4 - ye^z$

$h'(z) = 4 \qquad h(z) = 4z + k$

$\boxed{f = -\cos x + x\ln y - ye^z + 4z + k}$

$f = -\cos x + x\ln y - ye^{z} + 4z + K$

c) $\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

$= f(0,1,1) - f(0,1,0)$

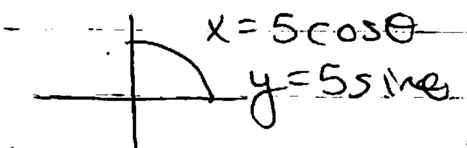
$= \cancel{-\cos 0} + 0\ln 1 - 1e^{1} + 4 - [\cancel{-\cos 0} + 0\ln 1 - 1e^{0}]$

$= -e + 4 + 1 = \boxed{5-e}$

d) $\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

$= \boxed{\cos x - \frac{x}{y^2} - ye^{z}}$

(15pts) 3. mass $= \int \rho(x,y)\,ds$




$= \int_0^{\pi/2} 5\cos\theta\, 5\sin\theta \sqrt{(-5\sin\theta)^2 + (5\cos\theta)^2}\, d\theta$

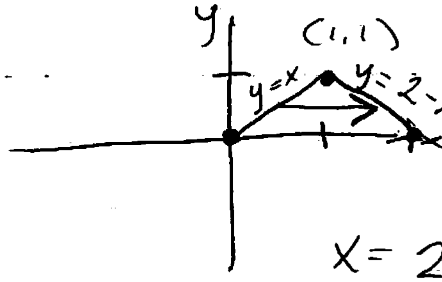
$= \int_0^{\pi/2} 5\cos\theta\, 5\sin\theta \sqrt{25}\, d\theta$

$u = 5\sin\theta \quad du = 5\cos\theta\, d\theta$

$= \int_0^{5} u5\, du = 5\frac{u^2}{2}\Big|_0^5 = \boxed{\frac{125}{2}}$

(12pts) 4. $P = 2x + \tan x \rightarrow \frac{\partial P}{\partial y} = 0$

$Q = x^2 + \sqrt{y} \rightarrow \partial Q/\partial x = 2x$




$m = \frac{0-1}{2-1} = -1$

$$\iint 2x - 0 \, dA$$

$x = 2 - y$

$$\int_0^1 \int_y^{2-y} 2x \, dx\, dy$$

$$\int_0^1 x^2 \Big|_y^{2-y} dy = \int_0^1 (2-y)^2 - y^2 \, dy$$

$$= \int_0^1 4 - 4y + \cancel{y^2} - \cancel{y^2} \, dy$$

$$4y - 2y^2 \Big|_0^1 = 4 - 2 = \boxed{2}$$

(13pts) 5. $\vec{F} = \langle P, Q, R \rangle$ $\quad\vec{F}$ conservative $\rightarrow$

$\vec{F} = \nabla f = \langle f_x, f_y, f_z \rangle$

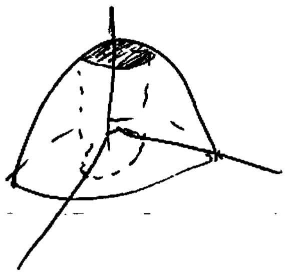
$\operatorname{curl} \vec{F} = \operatorname{curl} \nabla f = \nabla \times \nabla f$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left\langle \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, -\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right), \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right\rangle$$

$= \vec{0}$ by Clairaut's thm

$(f_{xy} = f_{yx})$

(15pts) 6.



$$|\vec{r}_x \times \vec{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$
$$= \sqrt{(-2x)^2 + (-2y)^2 + 1}$$
$$= \sqrt{4x^2 + 4y^2 + 1}$$

$$\iint_D \sqrt{4x^2+4y^2+1}\, dA = \int_0^{2\pi}\int_0^{1} \sqrt{4r^2+1}\, r\, dr\, d\theta$$

$u = 4r^2 + 1$
$du = 8r\,dr$
$\frac{1}{8} du = r\,dr$

$$\int_0^{2\pi}\int_1^{5} \frac{1}{8}\sqrt{u}\, du\, d\theta$$

$$\frac{2\pi}{8} \cdot \frac{2}{3} u^{3/2} \Big|_1^5 = \boxed{\frac{\pi}{6}\left[5^{3/2} - 1\right]}$$