MA 241 Test 4 Version 2

Be sure to specify every test you use and to fully justify your work as we have done in class.
No work=No Credit!

1. (12 points) Find the Taylor series for f(x)=cosx centered at $a=\pi$

2. (13 points) Find a power series representation for $f(x)=\ln(5+x^2)$ and determine its radius of convergence. Hint: You may want to find $f'(x)$

3. (10 points) Use the integral test to determine the convergence or divergence of $\sum_{n=2}^{\infty} \frac{1}{n[\ln(n)]^2}$

4. (20 points) Determine if the following series converges or diverges. If it converges, find its sum

a) $\sum_{n=1}^{\infty} \frac{1}{7n}$

b) $\sum_{n=0}^{\infty} \frac{(-3)^n}{4^{n+1}}$

c) $\sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3}$ Include the first 3 partial sums with your answer

5. (16 points) Find the radius and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(x+5)^n}{4^n n^3}$

Show all of your work.

6. (24 points) Determine if the following series converge or diverge.
Justify your answers thoroughly.

a) $\sum_{n=1}^{\infty} \frac{4}{\sqrt[3]{n+1}}$

b) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{5^{2n}}$

c) $\sum_{n=1}^{\infty} \sin\left(\frac{6n-1}{8n+1}\right)$

d) $\sum_{n=1}^{\infty} \frac{1}{4^n + \sqrt[5]{n}}$

7. (5 points) ***Briefly*** explain the difference between a sequence and a series.

C2 T4 V2 Solutions

1. (12 pts)

$f(x) = \cos x \qquad f(\pi) = -1$

$f'(x) = -\sin x \qquad f'(\pi) = 0$

$f''(x) = -\cos x \qquad f''(\pi) = 1$

$f'''(x) = \sin x \qquad f'''(\pi) = 0$

$f^{4}(x) = \cos x \qquad f^{4}(\pi) = -1$

$$\cos x = \sum_{n=0}^{\infty} \frac{f^{n}(\pi)(x-\pi)^n}{n!} = f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)(x-\pi)^2}{2!} + \dots$$

$$= -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \dots$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-\pi)^{2n}}{(2n)!}}$$

2. (13 pts) $f'(x) = \frac{2x}{5+x^2} = \frac{2x}{5}\left(\frac{1}{1+\frac{x^2}{5}}\right)$ $\boxed{R = \sqrt{5}}$

$$= \frac{2x}{5}\left(\frac{1}{1-\left(-\frac{x^2}{5}\right)}\right) \qquad \left|-\frac{x^2}{5}\right| < 1$$

$$= \frac{2x}{5}\sum_{n=0}^{\infty}\left(-\frac{x^2}{5}\right)^n = \frac{2x}{5}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{5^n}$$

$$= \sum_{n=0}^{\infty}\frac{2(-1)^n x^{2n+1}}{5^{n+1}}$$

$$f(x) = C + \sum_{n=0}^{\infty}\frac{2(-1)^n x^{2n+2}}{(2n+2)\,5^{n+1}}$$

$$f(0) = \ln(5+0^2) = C$$

$$\boxed{f(x) = \ln 5 + \sum_{n=0}^{\infty}\frac{2(-1)^n x^{2n+2}}{(2n+2)\,5^{n+1}}}$$

3. (10pts)

$$\int_2^\infty \frac{1}{x(\ln x)^2}\,dx \qquad u=\ln x \quad du=\frac{1}{x}dx$$

$$\int_{\ln 2}^\infty \frac{1}{u^2}\,du = \lim_{n\to\infty}\int_{\ln 2}^n \frac{1}{u^2}\,du$$

$$= \lim_{n\to\infty} -\frac{1}{u}\Big|_{\ln 2}^n = \lim_{n\to\infty} -\frac{1}{n} + \frac{1}{\ln 2} = \frac{1}{\ln 2}$$

Converges

$\sum \frac{1}{n(\ln n)^2}$ converges by Integral test

4. (20pts)

a) Harmonic series diverges

b) $$\sum_{n=0}^\infty \frac{(-3)^n}{4^{n+1}} = \frac{1}{4} - \frac{3}{4^2} + \frac{3^2}{4^3} - \frac{3^3}{4^4} + \cdots$$

$a + ar + ar^2 + \cdots$ Geometric series

$a = \frac{1}{4} \qquad r = \frac{-3}{4} \qquad \left|\frac{-3}{4}\right| < 1$ converges to

$$\frac{a}{1-r} = \frac{\frac{1}{4}}{1+\frac{3}{4}} = \frac{\frac{1}{4}}{\frac{7}{4}} = \boxed{\frac{1}{7}}$$

c) $$S_1 = \frac{1}{2} - \frac{1}{4}$$

$$S_2 = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5}$$

$$S_3 = \frac{1}{2} - \cancel{\frac{1}{4}} + \frac{1}{3} - \frac{1}{5} + \cancel{\frac{1}{4}} - \frac{1}{6}$$

Telescoping Sum

$$S_n = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$\lim_{n\to\infty} S_n = \boxed{\frac{1}{2} + \frac{1}{3}}$$ converges

5. (16pts) Find the radius & int of C

$$\lim_{n\to\infty}\left|\frac{(x+5)^{n+1}}{4^{n+1}(n+1)^3}\cdot\frac{4^n n^3}{(x+5)^n}\right| = \lim_{n\to\infty}\left|\frac{(x+5)\,n^3}{4\,(n+1)^3}\right|$$

$$= \left|\frac{x+5}{4}\right| < 1$$

$|x+5| < 4$ $\quad R=4 \quad a=-5$

Endpts $\quad x=a-R$, $\quad x=a+R$

$x=-9$, $\quad x=-1$

$x=-9$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-4)^n}{4^n n^3} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$ Alt series test

$C_n = \frac{1}{n^3}$ $\quad \frac{1}{n^3} > \frac{1}{(n+1)^3}$ decreasing ✓

$\lim_{n\to\infty}\frac{1}{n^3} = 0$ ✓

Converges

$x=-1$: $\displaystyle\sum_{n=1}^{\infty}\frac{4^n}{4^n n^3} = \sum_{n=1}^{\infty}\frac{1}{n^3}$ p-series $p=3>1$

Converges

$\boxed{\text{Interval of C} \quad [-9,-1]}$

6. (24pts)

a) $\lim_{n\to\infty} \dfrac{\frac{4}{\sqrt[3]{n+1}}}{\frac{1}{\sqrt[3]{n}}} = \lim_{n\to\infty} \dfrac{4\sqrt[3]{n}}{\sqrt[3]{n+1}} = 4 > 0$

$\sum \frac{1}{\sqrt[3]{n}}$ p-series $p = \frac{1}{3} < 1$ diverges

$\sum \frac{4}{\sqrt[3]{n+1}}$ diverges by LCT

b) Ratio test $\lim_{n\to\infty} \left| \frac{\sqrt{n+1}}{5^{2n+2}} \cdot \frac{5^{2n}}{\sqrt{n}} \right|$

$= \lim_{n\to\infty} \left| \frac{\sqrt{n+1}}{5^2 \sqrt{n}} \right| = \frac{1}{25} < 1$ converges

c) $\lim_{n\to\infty} \sin\left(\frac{6n-1}{8n+1}\right) = \sin\left(\frac{6}{8}\right) \neq 0$

diverges by Divergence test

d) $\sum_{n=1}^{\infty} \frac{1}{4^n + \sqrt{n}} \leq \sum_{n=1}^{\infty} \frac{1}{4^n}$ Geometric series

$r = \frac{1}{4}$ $\quad |\frac{1}{4}| < 1$

converges

converges by comparison test

7. (5pts) A Sequence is a list of numbers written in a definite order; a series is a sum of a sequence.