MA 241 Test 4 Version 1

Be sure to specify every test you use and to fully justify your work as we have done in class. No work=No Credit!

1. (12 points) Find the Taylor series for f(x)=sinx centered at $a=\pi$

2. (13 points) Find a power series representation for $f(x)=\ln(3+x^2)$ and determine its radius of convergence. Hint: You may want to find $f'(x)$

3. (10 points) Use the integral test to determine the convergence or divergence of $\sum_{n=2}^{\infty} \frac{1}{n[\ln(n)]^2}$

4. (20 points) Determine if the following series converges or diverges. If it converges, find its sum

a) $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2}$ Include the first 3 partial sums with your answer

b) $\sum_{n=0}^{\infty} \frac{(-2)^n}{5^{n+1}}$

c) $\sum_{n=1}^{\infty} \frac{1}{6n}$

5. (16 points) Find the radius and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(x+3)^n}{6^n n^4}$

Show all of your work.

6. (24 points) Determine if the following series converge or diverge. Justify your answers thoroughly.

a) $\sum_{n=1}^{\infty} \frac{1}{8^n + \sqrt[3]{n}}$

b) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{3^{2n}}$

c) $\sum_{n=1}^{\infty} \cos\left(\frac{3n+1}{5n-1}\right)$

d) $\sum_{n=1}^{\infty} \frac{6}{\sqrt[4]{n+1}}$

7. (5 points) ***Briefly*** explain the difference between a sequence and a series.

C2 T4 V1 Solutions

1. (12pts) $f(x) = \sin x$ $\quad f(\pi) = 0$

$f'(x) = \cos x \quad f'(\pi) = -1$

$f''(x) = -\sin x \quad f''(\pi) = 0$

$f'''(x) = -\cos x \quad f'''(\pi) = 1$

$f^4(x) = \sin x \quad f^4(\pi) = 0$

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi)(x-\pi)^n}{n!} = f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)(x-\pi)^2}{2!} + \frac{f'''(\pi)(x-\pi)^3}{3!} + \cdots$$

$$= -(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!} + \cdots$$

$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-\pi)^{2n+1}}{(2n+1)!}}$$

2. (13pts) $f'(x) = \dfrac{2x}{3+x^2} = \dfrac{2x}{3}\left(\dfrac{1}{1+\frac{x^2}{3}}\right)$

$$= \frac{2x}{3}\left(\frac{1}{1-\left(-\frac{x^2}{3}\right)}\right)$$

$$= \frac{2x}{3} \sum_{n=0}^{\infty} \left(-\frac{x^2}{3}\right)^n \qquad \left|-\frac{x^2}{3}\right| < 1 \qquad \boxed{R = \sqrt{3}}$$

$$= \frac{2x}{3} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{3^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}\, 2}{3^{n+1}}$$

$$f(x) = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}\, 2}{(2n+2)\, 3^{n+1}}$$

$$\ln(3+0^2) = C$$

$$\boxed{\ln(3+x^2) = \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}\, 2}{(2n+2)\, 3^{n+1}}}$$

3. (10pts)

$$\int_2^\infty \frac{1}{x(\ln x)^2}\,dx \qquad u=\ln x \quad du=\frac{1}{x}\,dx$$

$$\int_{\ln 2}^\infty \frac{1}{u^2}\,du = \lim_{n\to\infty}\int_{\ln 2}^{n} \frac{1}{u^2}\,du$$

$$= \lim_{n\to\infty} -\frac{1}{u}\Big|_{\ln 2}^{n} = \lim_{n\to\infty} -\frac{1}{n} + \frac{1}{\ln 2} = \frac{1}{\ln 2}$$

Converges

$\sum \frac{1}{n(\ln n)^2}$ Converges by the Integral test.

4. (20pts)

a)

$$\boxed{\begin{aligned} S_1 &= 1-\tfrac{1}{3} \\ S_2 &= 1-\tfrac{1}{3}+\tfrac{1}{2}-\tfrac{1}{4} \\ S_3 &= 1-\cancel{\tfrac{1}{3}}+\tfrac{1}{2}-\tfrac{1}{4}+\cancel{\tfrac{1}{3}}-\tfrac{1}{5} \end{aligned}}$$

Telescoping Sum

$$S_n = 1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}$$

$$\lim_{n\to\infty} S_n = 1+\frac{1}{2} = \boxed{\frac{3}{2}}$$ Converges

b) $$\sum_{n=0}^{\infty}\frac{(-2)^n}{5^{n+1}} = \frac{1}{5}-\frac{2}{5^2}+\frac{2^2}{5^3}-\cdots$$

$a+ar+ar^2+\cdots$ Geometric series

$a=\frac{1}{5}$ $r=-\frac{2}{5}$ $\left|-\frac{2}{5}\right|<1$ Converges to $\frac{\frac{1}{5}}{1+\frac{2}{5}} = \frac{\frac{1}{5}}{\frac{7}{5}} = \boxed{\frac{1}{7}}$

c) Harmonic series diverges

5. (16pts)

Ratio Test $\lim_{n\to\infty}\left|\frac{(x+3)^{n+1}}{6^{n+1}(n+1)^4}\cdot\frac{6^n n^4}{(x+3)^n}\right| = \lim_{n\to\infty}\left|\frac{(x+3)\,n^4}{6(n+1)^4}\right|$

$= \left|\frac{x+3}{6}\right| < 1$

$|x+3| < 6$

$\boxed{R=6}$ $\quad a=-3$

Endpts $\quad x=a-R \;,\; x=a+R$

$x=-9 \;,\; x=3$

$x=-9$: $\sum_{n=1}^{\infty}\frac{(-6)^n}{6^n n^4} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$ Alternating Series Test

$c_n = \frac{1}{n^4}$ $\quad \frac{1}{n^4} > \frac{1}{(n+1)^4}$ decreasing ✓

$\lim_{n\to\infty}\frac{1}{n^4} = 0$ ✓ converges

$x=3$: $\sum_{n=1}^{\infty}\frac{6^n}{6^n n^4} = \sum_{n=1}^{\infty}\frac{1}{n^4}$ p-series $p=4>1$

Converges

Interval of C

$[-9, 3]$

6. (24pts)

a) $\sum_{n=1}^{\infty} \frac{1}{8^n + \sqrt[3]{n}} \leq \sum_{n=1}^{\infty} \frac{1}{8^n}$ $r = \frac{1}{8}$ $\left|\frac{1}{8}\right| < 1$

Converges Geometric series

converges by Comparison test

b) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{3^{2n}}$ Ratio Test $\lim_{n\to\infty} \left| \frac{\sqrt{n+1}}{3^{2n+2}} \cdot \frac{3^{2n}}{\sqrt{n}} \right|$

$= \lim_{n\to\infty} \left| \frac{\sqrt{n+1}}{\sqrt{n}\, 3^2} \right| = \frac{1}{9} < 1$

Converges

c) $\lim_{n\to\infty} \cos\left(\frac{3n+1}{5n-1}\right) = \cos\left(\frac{3}{5}\right) \neq 0$

diverges divergence test

d) $\sum_{n=1}^{\infty} \frac{6}{\sqrt[4]{n+1}}$

$\lim_{n\to\infty} \frac{\frac{6}{\sqrt[4]{n+1}}}{\frac{1}{\sqrt[4]{n}}} = \lim_{n\to\infty} \frac{6\sqrt[4]{n}}{\sqrt[4]{n+1}} = 6 > 0$

$\sum \frac{1}{\sqrt[4]{n}}$ p-series $p = \frac{1}{4} < 1$ diverges

$\sum \frac{6}{\sqrt[4]{n+1}}$ diverges by LCT

7. (5pts) A sequence is a list of numbers written in a definite order; a series is a sum of a sequence.