Calculus II Test 3 Version 1

1. (12 points) An isolated population is limited by its food supply to 4,000 individuals. The current population is 2,000 individuals; 40 years ago (t=0) it was 1,000 individuals. Use the Logistic Equation to find a formula for P(t) the population at time t.

2. (17 points) Solve the Initial Value Problem $y''+2y'+10y=50e^{3x}$; $y(0)=3$, $y'(0)=-4$

3. (9 points) Find the general solution of $y''-4y'+4y=0$

4. (15 points) Use the differential equation $y''+y'-6y=f(x)$ along with the value of f(x) listed below to answer the following.

a) Find the complementary solution y_c

b) Find the form of y_p, but do **NOT** solve for the coefficients.

i) $f(x)=xe^{-x}$

ii) $f(x)=6\cos(3x)$

iii) $f(x)=e^{2x}+2x$

5. (15 points) A metal ball with a weight of 6 lb stretches a spring 3 inches. The damping constant is 2 and the spring is compressed 4 inches from its equilibrium and released with zero velocity. HINT : Gravity is 32 ft / s^2

a) If x(t) is the position of the metal ball at time t, formulate (**do NOT solve**) the IVP that describes the motion of the ball.

b) Determine what kind of damping this is and show your work. Do NOT multiply the numbers out!

6. (12 points) A cup of hot chocolate is 50°C when it is first made and 40°C 10 minutes later. The room it is sitting in is 30°C, find the temperature 20 minutes after it was made. Hint: $T(t)=T_s+(T_0-T_s)e^{-kt}$

7. (8 points) Determine if the following sequences converge or diverge. If a sequence converges, find its limit. You do not need to show work.

a) $a_n=\dfrac{8^n}{3^{n+5}}$

b) $a_n=\dfrac{-19n^3+77n^2+2}{1-10n^2+3n^3}$

8. (5 points) What does it mean if a sequence is monotonic? (Keep your answers brief)

9.(7 points) Use the Squeeze Theorem to determine the convergence of $a_n=\dfrac{(-1)^n n}{n^2+n}$

C2 T3 V1 Solutions

1. (12pts) $P(t) = \frac{C}{1+Ae^{-kt}}$

$P(t) = \frac{4000}{1+Ae^{-kt}}$

$P(40) = 2000$

$P(0) = 1000$

$$P(0) = 1000 = \frac{4000}{1+A}$$

$$1+A = 4$$

$$A = 3$$

$$P(40) = 2000 = \frac{4000}{1+3e^{-k40}}$$

$$1+3e^{-k40} = 2$$

$$3e^{-k40} = 1$$

$$e^{-k40} = \frac{1}{3}$$

$$-k40 = \ln\left(\frac{1}{3}\right)$$

$$-k = \frac{1}{40}\ln\left(\frac{1}{3}\right)$$

$$P(t) = \frac{4000}{1+3e^{\frac{1}{40}\ln\left(\frac{1}{3}\right)t}}$$

2. (17pts) $y'' + 2y' + 10y = 0$

$r^2 + 2r + 10 = 0$

$r = \frac{-2 \pm \sqrt{4-40}}{2} = \frac{-2 \pm \sqrt{-36}}{2} = \frac{-2 \pm 6i}{2}$

$= -1 \pm 3i$

$y_c = e^{-x}[C_1 \cos 3x + C_2 \sin 3x]$

$y_p = Ae^{3x} \quad y_p' = 3Ae^{3x} \quad y_p'' = 9Ae^{3x}$

$9Ae^{3x} + 2(3Ae^{3x}) + 10Ae^{3x} = 50e^{3x}$

$25Ae^{3x} = 50e^{3x}$

$A = 2$

$y_p = 2e^{3x}$

$y = e^{-x}[C_1 \cos 3x + C_2 \sin 3x] + 2e^{3x}$

$y(0) = C_1 + 2 = 3 \longrightarrow C_1 = 1$

$y = e^{-x}[\cos 3x + C_2 \sin 3x] + 2e^{3x}$

$y' = -e^{-x}[\cos 3x + C_2 \sin 3x] + e^{-x}[-3\sin 3x + 3C_2 \cos 3x] + 6e^{3x}$

$y'(0) = -1 + 3C_2 + 6 = -4$

$3C_2 = -9$

$C_2 = -3$

$$\boxed{y = e^{-x}[\cos 3x - 3\sin 3x] + 2e^{3x}}$$

6. (12pts)

$T(t) = 30 + (50-30)e^{-k(10)}$

$T(10) = 40 = 30 + 20e^{-k(10)}$

$10 = 20e^{-k(10)}$

$\frac{1}{2} = e^{-k(10)}$

$\ln(\frac{1}{2}) = -10k$

$\frac{1}{10}\ln(\frac{1}{2}) = -k$

$T = 30 + 20e^{\frac{1}{10}\ln(\frac{1}{2})t}$

$T(20) = 30 + 20e^{\frac{1}{10}\ln(\frac{1}{2})20}$

7. a) $a_n = \frac{8^n}{3^n 3^5} = \left(\frac{8}{3}\right)^n \frac{1}{3^5}$

$\lim_{n\to\infty} a_n = \infty$ diverges

b) $\lim_{n\to\infty} a_n = \frac{-19}{3}$ converges

8. (5pts) A sequence is monotonic if it is always increasing or if it is always decreasing.

9. (7pts) $\frac{-n}{n^2+n} \le \frac{(-1)^n n}{n^2+n} \le \frac{n}{n^2+n}$

$\lim_{n\to\infty} \frac{-n}{n^2+n} = 0 = \lim_{n\to\infty} \frac{n}{n^2+n}$

$\lim_{n\to\infty} \frac{(-1)^n n}{n^2+n} = 0$ by Squeeze Thm.

3. (9 pts) $y'' - 4y' + 4y = 0$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$$y = C_1 e^{2x} + C_2 x e^{2x}$$

4. (15 pts) $y'' + y' - 6y = f(x)$

a) $r^2 + r - 6 = 0$

$(r+3)(r-2) = 0$

$$y_c = C_1 e^{-3x} + C_2 e^{2x}$$

b) i) $y_p = (Ax + B) e^{-x}$

ii) $y_p = A \cos 3x + B \sin 3x$

iii) $y_p = Ae^{2x} x + Bx + C$

5. (15 pts)

a) $mx'' + bx' + kx = 0$

$F = mg$

$6 = m(32)$

$m = \frac{6}{32} = \frac{3}{16}$

$F = kx$ ← $3\text{in} = \frac{3}{12}\text{ft}$

$6 = k(\frac{1}{4})$

$k = 24$

$$\frac{3}{16}x'' + 2x' + 24x = 0 \qquad x(0) = -\frac{1}{3} \qquad x'(0) = 0$$

b) $b^2 - 4mk = 2^2 - 4(\frac{3}{16})(24) < 0$ underdamping