MA 241 Test 2 Version 1

1. (10 points) Find the length of the curve represented by $x = \frac{1}{4}t^2 - \frac{1}{2}\ln(t),\ y = 10,\ \ 1 \le t \le 4$

Hint: First write the formula and then plug in.

2. (8 points) Find the average value of the function $y = \frac{1}{x^2}$ on $[1,3]$

3. (10 points) Write out the formulas needed to find the center of mass of the lamina bounded by $y = f(x)$, the x-axis, $x = a$, and $x = b$.

4. (12 points) A spring has a natural length of 1 ft. If it takes 10 lbs of force to maintain the spring stretched to 3 ft, find the work needed to stretch the spring from 3 ft to 6 ft. Include the appropriate units for your answer.

5. (12 points) Find the orthogonal trajectories of $y = kx^4$

6. (12 points) Use Euler's method with a stepsize of 0.5 to estimate y(1.5) and y(2) for the initial value problem $y' = 2(x + y),\ y(1) = 4$.

7. (12 points) A tank has the shape of an upright circular cylinder with height 11 ft and radius of 4 ft. The water level starts 2 ft down. Set up an integral (**DO NOT EVALUATE**) to find the work required to pump all of the water to a spout located 1 foot above the top of the tank. Use 62.5 lb/ft^3 for the weight density of water and include the appropriate units for the answer.

8. (12 points) A tank contains 70 liters of pure water. A solution containing 2 kg of salt per liter enters the tank at a rate of 7 liters/min. The solution is mixed and drains from the tank at the same rate. If y represents the amount of salt at time t, find an equation to represent the amount of salt in the tank after t minutes.

9. (12 points) The top of a vertical plate lies 10 m under the surface of the water. If the plate is a semicircle with radius 2 m with the flat end of the semicircle on the ground, set up an integral (**DO NOT EVALUATE**) to find the hydrostatic force against the gate. The density of water is 1000 kg/m^3 and gravity is $9.8 m/s^2$

C2 T2 V1 Solutions

(10pts) 1. $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= \int_1^4 \sqrt{\left(\frac{1}{2}t - \frac{1}{2}\frac{1}{t}\right)^2 + 0^2}\, dt$$

$$= \int_1^4 \frac{1}{2}t - \frac{1}{2}\frac{1}{t}\, dt$$

$$= \frac{1}{4}t^2 - \frac{1}{2}\ln|t| \Big|_1^4$$

$$= \frac{1}{4}4^2 - \frac{1}{2}\ln 4 - \left(\frac{1}{4} - \cancel{\frac{1}{2}\ln 1}\right)$$

$$= \boxed{\frac{15}{4} - \frac{1}{2}\ln 4}$$

2. (8pts) $f_{aver} = \frac{1}{b-a}\int_a^b f(x)\,dx$

$$= \frac{1}{3-1}\int_1^3 \frac{1}{x^2}\,dx$$

$$\frac{1}{2}\left(-\frac{1}{x}\right)\Big|_1^3$$

$$\frac{1}{2}\left(-\frac{1}{3} + 1\right) = \frac{1}{2}\left(\frac{2}{3}\right) = \boxed{\frac{1}{3}}$$

3. (10pts)

$$(\bar{x}, \bar{y}) = \left(\frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx}, \frac{\int_a^b \frac{1}{2}[f(x)]^2\,dx}{\int_a^b f(x)\,dx}\right)$$

4 (12 pts). $F = kx$

$$10 = k(2)$$

$$k = 5$$

$$W = \int_2^5 5x\,dx = \frac{5}{2}x^2 \Big|_2^5$$

$$= \frac{5}{2}\left[25 - 4\right] = \boxed{\frac{5}{2}(21) \text{ ft-lbs}}$$

5. (12 pts) $\frac{dy}{dx} = 4kx^3$

$$\perp: \frac{dy}{dx} = \frac{-1}{4kx^3}$$

Eliminate k: $k = \frac{y}{x^4}$

$$\frac{dy}{dx} = -\frac{1}{4\left(\frac{y}{x^4}\right)x^3} = -\frac{1}{\frac{4y}{x}}$$

$$\frac{dy}{dx} = \frac{-x}{4y}$$

$$\int y\,dy = \int -\frac{x}{4}\,dx$$

$$\boxed{\frac{1}{2}y^2 = -\frac{x^2}{8} + C}$$

or $\boxed{\frac{1}{2}y^2 + \frac{x^2}{8} = C}$

6. (12pts) $x_0 = 1 \quad y_0 = 4$

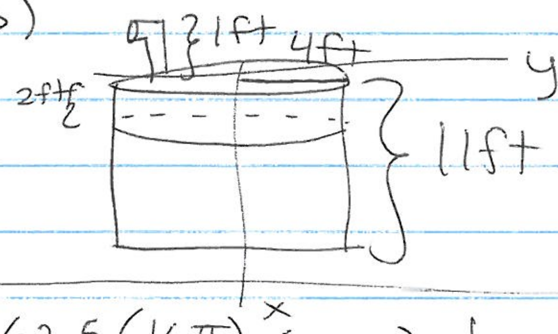
$x_1 = 1.5$

$x_2 = 2$

$$\begin{aligned} y(1.5) \approx y_1 &= y_0 + hF(x_0, y_0) \\ &= 4 + \tfrac{1}{2}F(1,4) \\ &= 4 + \cancel{\tfrac{1}{2}}(\cancel{2}(1+4)) \\ &= 9 \end{aligned}$$

$$\begin{aligned} y(2) \approx y_2 &= y_1 + hF(x_1, y_1) \\ &= 9 + \tfrac{1}{2}F(1.5, 9) \\ &= 9 + \cancel{\tfrac{1}{2}}(\cancel{2}(1.5+9)) \\ &= 19.5 \end{aligned}$$

7. (12pts)



$$\boxed{W = \int_2^{11} 62.5\underbrace{(16\pi)}_{\pi r^2}(x+1)\,dx \quad \text{ft-lbs}}$$

8 (12 pts)

$$\frac{dy}{dt} = (7)(2) - (7)\frac{y}{70} \qquad y(0) = 0$$

$$\frac{dy}{dt} = 14 - \frac{y}{10} = -\frac{1}{10}(y - 140)$$

$$\int \frac{dy}{y-140} = \int -\frac{1}{10}\,dt$$

$$\ln|y - 140| = -\frac{1}{10}t + C$$

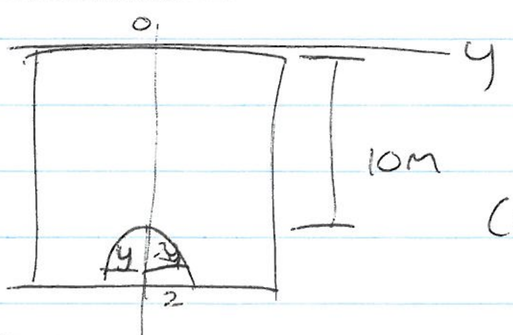
$$y - 140 = ke^{-t/10}$$

$$y = 140 + ke^{-t/10}$$

$$y(0) = 0 = 140 + ke^{0} \longrightarrow k = -140$$

$$\boxed{y = 140 - 140e^{-t/10}}$$

9. (12 pts)



$L = 2y$

$(x-12)^2 + (y-0)^2 = 4$

$y = \sqrt{4-(x-12)^2}$

$$\boxed{F = \int_{10}^{12} 1000(9.8)\, 2\sqrt{4-(x-12)^2}\; x\,dx \text{ N}}$$