MA 241 H Test 4 Be sure to show ALL of your work and specify every test you use.

1. (30 points) Determine if the following series converge or diverge. Find the sum of the convergent series. Justify your answers thoroughly as we have done in class.

a) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots$

b) $\sum_{n=0}^{\infty}\frac{2^n}{(-6)^{n+1}}$

c) $\sum_{n=1}^{\infty}\frac{1}{n+4}-\frac{1}{n+1}$ Your answer should include the first four partial sums.

d) $\sum_{n=1}^{\infty}e^{\left(\frac{1}{n^4}\right)}$

2. (15 points) a) Find a power series representation for $\frac{x^2}{4+x^7}$

b) Find its radius of convergence

c) Use your work from part a) to find $\int\frac{x^2}{4+x^7}\,dx$

3. (10 points) Use the integral test to determine the convergence or divergence of the following series $\sum_{n=0}^{\infty}2ne^{-n^2}$

4. (20 points) Find the radius and interval of convergence of the series $\sum_{n=1}^{\infty}\frac{(x-2)^n}{5^n\left(n^2+1\right)}$

5. (14 points) Determine if the following series converge or diverge
Justify your answers thoroughly.

a) $\sum_{n=1}^{\infty}\frac{(-5)^n 3^{n+1}}{(2n)!}$

b) $\sum_{n=1}^{\infty}\frac{3(-1)^n+8}{\sqrt[5]{n}}$

6. (6 points) If $\frac{1}{x}=\sum_{n=0}^{\infty}(-1)^n(x-1)^n$, find the sum of $\sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^{n+1}}{n+1}$

7. (5 points) Find the value of c such that $\sum_{n=0}^{\infty}e^{nc}=10$

C2 H T4 Solutions

1. (30pts)

a) $\sum_{n=1}^{\infty} \frac{1}{n}$ Harmonic diverges

b) $\sum_{n=0}^{\infty} \frac{2^n}{(-6)^{n+1}} = -\frac{1}{6} + \frac{2}{(-6)^2} + \frac{2^2}{(-6)^3} + \ldots$

Geometric

$a = -1/6$

$r = 2/-6 \quad |-2/6| < 1$

Converges to $\frac{a}{1-r} = \frac{-1/6}{1+2/6} = \frac{-\frac{1}{6}}{\frac{8}{6}} = \boxed{-\frac{1}{8}}$

c) $S_1 = \frac{1}{5} - \frac{1}{2}$

$S_2 = \frac{1}{5} - \frac{1}{2} + \frac{1}{6} - \frac{1}{3}$

$S_3 = \frac{1}{5} - \frac{1}{2} + \frac{1}{6} - \frac{1}{3} + \frac{1}{7} - \frac{1}{4}$

$S_4 = \cancel{\frac{1}{5}} - \frac{1}{2} + \frac{1}{6} - \frac{1}{3} + \frac{1}{7} - \frac{1}{4} + \frac{1}{8} - \cancel{\frac{1}{5}}$

$S_5 = -\frac{1}{2} + \cancel{\frac{1}{6}} - \frac{1}{3} + \frac{1}{7} - \frac{1}{4} + \frac{1}{8} + \frac{1}{9} - \cancel{\frac{1}{6}}$

$S_n = -\frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4}$

$\lim_{n\to\infty} S_n = \boxed{-\frac{1}{2} - \frac{1}{3} - \frac{1}{4}}$ converges

d) $\sum_{n=1}^{\infty} e^{\frac{1}{n^4}}$

Divergence test

$\lim_{n\to\infty} e^0 = 1 \neq 0$ diverges

2. (15 pts)

a) $\frac{x^2}{4+x^7} = \frac{x^2}{4}\left(\frac{1}{1+\frac{x^7}{4}}\right) = \frac{x^2}{4}\left(\frac{1}{1-\left(-\frac{x^7}{4}\right)}\right)$

$= \frac{x^2}{4} \sum_{n=0}^{\infty} \left(-\frac{x^7}{4}\right)^n$

$= \frac{x^2}{4} \sum_{n=0}^{\infty} \frac{(-1)^n x^{7n}}{4^n}$

$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{7n+2}}{4^{n+1}}}$

$\left|-\frac{x^7}{4}\right| < 1$

$|x^7| < 4$

$|x| < 4^{1/7}$

b) $R = 4^{1/7}$

c) $\int \sum_{n=0}^{\infty} \frac{(-1)^n x^{7n+2}}{4^{n+1}}\, dx$

$\boxed{C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{7n+3}}{4^{n+1}(7n+3)}}$

3. (10pts)

$$\int_0^\infty \underline{2x}e^{-x^2}\underline{dx} =$$

$u = -x^2 \qquad u(0) = 0$

$du = \underline{-2x\,dx} \qquad u(\text{"}\infty\text{"}) = -\text{"}\infty\text{"}$

$$= \int_0^{-\infty} -e^u\,du = \int_{-\infty}^0 e^u\,du$$

$$= \lim_{t\to-\infty} \int_t^0 e^u\,du$$

$$= \lim_{t\to-\infty} e^u \Big|_t^0$$

$$= \lim_{t\to-\infty} e^0 - \cancel{e^t} = \boxed{1} \leftarrow \text{fixed \& finite}$$

Converges by Integral

4. (20pts)

Ratio test $\displaystyle\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{5^{n+1}((n+1)^2+1)}\cdot\frac{5^n(n^2+1)}{(x-2)^n}\right|$

$$= \lim_{n\to\infty}\left|\frac{(x-2)(n^2+1)}{5((n+1)^2+1)}\right| = \left|\frac{x-2}{5}\right| < 1$$

$$|x-2| < 5$$

$\boxed{R=5} \qquad a=2$

$x = 2 - 5 = -3 \qquad x = 2 + 5 = 7$

$x = -3$

$$\sum_{n=1}^{\infty} \frac{(-5)^n}{5^n(n^2+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2+1)}$$

Alt series test

$$c_n = \frac{1}{n^2+1}$$

$$\frac{1}{n^2+1} > \frac{1}{(n+1)^2+1} \quad \text{decreasing } \checkmark$$

$$\lim_{n\to\infty} \frac{1}{n^2+1} = 0$$

(converge)

$$x = 7 \quad \sum_{n=1}^{\infty} \frac{5^n}{5^n(n^2+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

p-series
$p = 2 > 1$
converges

converges by comparison

(LCT also works, as does integral)

Interval of C

$[-3, 7]$

5. (14 pts)

a) Ratio $\lim_{n\to\infty} \left| \frac{5^{n+1}\, 3^{n+2}}{(2n+2)!} \cdot \frac{(2n)!}{5^n\, 3^{n+1}} \right|$

$= \lim_{n\to\infty} \left| \frac{5 \cdot 3}{(2n+2)(2n+1)} \right| = 0 < 1$

Converges

b) $\sum_{n=1}^{\infty} \frac{3(-1)^n + 8}{\sqrt[5]{n}} \geq \sum_{n=1}^{\infty} \frac{5}{\sqrt[5]{n}}$

diverges by comparison

p-series $p = 1/5 < 1$ diverges

6. (6 pts)

$\int \frac{1}{x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n (x-1)^n \, dx$

$\ln|x| + C = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}$

$x = 1 \rightarrow C = 0$

$\boxed{\ln|x|}$

7. (5 pts) $\sum_{n=0}^{\infty} e^{nc} = 1 + e^c + e^{2c} + e^{3c} + \cdots$

$a = 1 \quad r = e^c \quad \frac{a}{1-r} = 10 \quad \frac{1}{1-e^c} = 10 \quad \frac{1}{10} = 1 - e^c$

$e^c = 9/10$

$\boxed{c = \ln(9/10)}$