MA 241 H Test 4

Be sure to specify every test you use and to fully justify your work as we have done in class.
No work=No Credit!

1. (12 points) Find the Taylor series for f(x)=sinx centered at a=π

2. (13 points) Find a power series representation for $f(x)=\ln(3+x^2)$ and determine its radius of convergence

3. (10 points) Use the integral test to determine the convergence or divergence of $\sum_{n=2}^{\infty}\frac{1}{n[\ln(n)]^2}$

4. (20 points) Determine if the following series converges or diverges. If it converges, find its sum

a) $\sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n+2}$ Include the first 3 partial sums with your answer as well as s_n

b) $\sum_{n=0}^{\infty}\frac{(-2)^n}{5^{n+1}}$

c) $\sum_{n=1}^{\infty}\frac{1}{6n}$

5. (16 points) Find the radius and interval of convergence of the series $\sum_{n=1}^{\infty}\frac{(x+3)^n}{6^n n^4}$

Show all of your work.

6. (24 points) Determine if the following series converge or diverge.
Justify your answers thoroughly.

a) $\sum_{n=1}^{\infty}\frac{1}{8^n+\sqrt[3]{n}}$

b) The series $\sum a_n$ defined by the equations

$$a_1=2 \quad a_{n+1}=\frac{\sqrt{n}}{3^{2n}}a_n$$

c) $\sum_{n=1}^{\infty}\tan\left(\frac{3n+1}{5n-1}\right)$

d) $\sum_{n=1}^{\infty}\frac{6}{\sqrt[4]{n+1}}$

7. (5 points) ***Briefly*** explain the difference between a sequence and a series.

C2 H T4 Solutions

1. (12pts) $f(x)=\sin x \qquad f(\pi)=0$

$f'(x)=\cos x \qquad f'(\pi)=-1$

$f''(x)=-\sin x \qquad f''(\pi)=0$

$f'''(x)=-\cos x \qquad f'''(\pi)=1$

$f^4(x)=\sin x \qquad f^4(\pi)=0$

$$\sin x = \sum_{n=0}^{\infty} \frac{f^n(\pi)(x-\pi)^n}{n!} = f(\pi)+f'(\pi)(x-\pi)+\frac{f''(\pi)(x-\pi)^2}{2!}+\cdots$$

$$= -(x-\pi)+\frac{(x-\pi)^3}{3!}-\frac{(x-\pi)^5}{5!}+\cdots$$

$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-\pi)^{2n+1}}{(2n+1)!}}$$

2. (13pts) $f(x)=\ln(3+x^2)$

$$f'(x)=\frac{2x}{3+x^2}=\frac{2x}{3\left(1-\left(-\frac{x^2}{3}\right)\right)}$$

$$=\frac{2x}{3}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{3^n}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}2}{3^{n+1}}$$

$$\boxed{R=\sqrt{3}}$$

$$f(x)=\int \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}2}{3^{n+1}}\,dx$$

$$\ln(3+x^2)=C+\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}2}{3^{n+1}(2n+2)}$$

$x=0$

$\ln 3 = C$

$$\ln(3+x^2)=\boxed{\ln 3+\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}2}{3^{n+1}(2n+2)}}$$

3. (10pts) $\int_2^\infty \frac{1}{x(\ln x)^2}dx$ $\quad u=\ln x$

$du=\frac{1}{x}$

$$\int_{\ln 2}^\infty \frac{1}{u^2}du = \lim_{t\to\infty}\int_{\ln 2}^t \frac{1}{u^2}du$$

$$=\lim_{t\to\infty} -\frac{1}{u}\Big|_{\ln 2}^t$$

$$=\lim_{t\to\infty} -\frac{1}{t}+\frac{1}{\ln 2}=\frac{1}{\ln 2} \quad \text{Converges}$$

$\sum \frac{1}{n(\ln n)^2}$ Converges by Integral test

4. (20pts)

a) $S_1 = 1-\frac{1}{3}$

$S_2 = 1-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}$

$S_3 = 1-\cancel{\frac{1}{3}}+\frac{1}{2}-\frac{1}{4}+\cancel{\frac{1}{3}}-\frac{1}{5}$

$S_n = 1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}$

$\lim_{n\to\infty} S_n = \boxed{1+\frac{1}{2}}$ converges Telescoping

b) $\sum_{n=0}^\infty \frac{(-2)^n}{5^{n+1}} = \frac{1}{5} - \frac{2}{5^2} + \frac{2^2}{5^3} - \cdots$

$a \quad +ar \quad +ar^2$

Geometric series

$a=\frac{1}{5}$ $\quad r=\frac{-2}{5}$ $\quad \left|-\frac{2}{5}\right|<1$ converges to $\frac{1/5}{1+\frac{2}{5}}=\frac{\frac{1}{5}}{\frac{7}{5}}=\boxed{\frac{1}{7}}$

c) Harmonic series diverges

5. (16pts)

Ratio test $\displaystyle\lim_{n\to\infty}\left|\frac{(x+3)^{n+1}}{6^{n+1}(n+1)^4}\cdot\frac{6^n n^4}{(x+3)^n}\right| = \lim_{n\to\infty}\left|\left(\frac{(x+3)\,n^4}{6(n+1)^4}\right)\right|$

$\cancel{\lim_{n\to\infty}}\ \left|\frac{x+3}{6}\right| < 1 \qquad |x+3| < 6$

$\boxed{R=6}$

endpts $x = a-R$, $x = a+R$

$x = -3-6$ $\qquad x = 3$

$x = -9$

$x=-9$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-6)^n}{6^n n^4} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$

Alt series test

$c_n = \frac{1}{n^4} > \frac{1}{(n+1)^4}$ dec ✓

$\lim_{n\to\infty}\frac{1}{n^4} = 0$ ✓

converges

$x=3$: $\displaystyle\sum_{n=1}^{\infty}\frac{6^n}{6^n n^4} = \sum_{n=1}^{\infty}\frac{1}{n^4}$ $\qquad$ p-series $p=4>1$ converges

$\boxed{\text{I of C: } [-9, 3]}$

6. (24pts)

a) $\sum_{n=1}^{\infty} \frac{1}{8^n+\sqrt[3]{n}} \leq \sum_{n=1}^{\infty} \frac{1}{8^n}$ Geometric series

$r=\frac{1}{8}$ $\left|\frac{1}{8}\right|<1$ converges

converges
comparison
test

b) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\sqrt{n}\ \cancel{a_n}}{3^{2n}\ \cancel{a_n}}\right| = 0 < 1$ converges

Ratio test

c) $\lim_{n\to\infty} \tan\left(\frac{3n+1}{5n-1}\right) = \tan\left(\frac{3}{5}\right) \neq 0$ diverges

Divergence test

d) Limit Comparison test $\lim_{n\to\infty} \frac{\frac{6}{4\sqrt{n+1}}}{\frac{1}{\sqrt[4]{n}}} = \lim_{n\to\infty} \frac{6\sqrt[4]{n}}{4\sqrt{n+1}} = 6>0$

$\sum \frac{1}{\sqrt[4]{n}}$ p-series
$p=\frac{1}{4}<1$ diverges

$\sum \frac{1}{\sqrt[4]{n+1}}$ diverges by LCT

7. (5pts) A sequence is a list of numbers in a definite order; a series is the sum of a sequence.