MA 241 Honors Test 3

1. (12 points) Provided that the temperature is constant, the rate of change of atmospheric pressure P with respect to altitude h is proportional to P. At a certain fixed temperature, the pressure is 100 kPa at sea level and 10 kPa at $h = 1000$m. For this temperature, find the pressure P at altitude h.

2. (10 points) Solve the IVP $y'' + 8y' + 16y = 0$; $y(0) = 0$, $y'(0) = 12$

3. (15 points) Find the general solution $y'' + 2y' + 5y = 60e^{3x}$

4. (18 points) Use the differential equation $y'' + 5y' + 6y = f(x)$ along with the value of f(x) listed below to find the form of the particular solution, y_p, but do **NOT** solve for the coefficients.

a) $f(x) = e^{-3x}$

b) $f(x) = 11x^2$

c) $f(x) = \sin(6x) + 4e^{-2x}$

d) $f(x) = e^{-2x}\sin(6x)$

5. (12 points) Prove that xe^{rx} is a solution to $ay'' + by' + cy = 0$ if r is a repeated root

6. (17 points) A mass with a weight of 7 lb stretches a spring 3 inches. The damping constant is 5. The spring is compressed 7 inches from its equilibrium position and released with no velocity.

a) If x(t) is the position of the mass at time t, formulate the IVP that describes the motion of the mass.
HINT : Acceleration due to gravity is 32 ft/s^2

b) What kind of damping is this? Justify your answer.

7. (6 points) Find the general form of the sequence $\left\{-3, \frac{4}{3}, \frac{-5}{9}, \frac{6}{27}, \ldots\right\}$

8. (10 points) Use $a_n = \frac{(-2)^{n+1}}{3^n}$ to answer the following. Justify your answers.

a) Determine if the sequence $\{a_n\}$ converges or diverges. If it converges, find its limit.

b) Is this sequence monotonic? Briefly explain your answer

C2 H T3 Solutions

1. (12pts) $\frac{dP}{dh} = kP$

$$P(h) = P_0 e^{kh}$$

$$P_0 = 100$$

$$P(1000) = 10 = 100e^{k1000}$$

$$\frac{1}{10} = e^{k1000}$$

$$\ln\left(\frac{1}{10}\right) = k1000$$

$$k = \frac{1}{1000}\ln\left(\frac{1}{10}\right)$$

$$\boxed{P(h) = 100e^{\frac{h}{1000}\ln\left(\frac{1}{10}\right)}}$$

2. (10pts) $y'' + 8y' + 16y = 0 \quad y(0) = 0 \quad y'(0) = 12$

$$r^2 + 8r + 16 = 0$$

$$(r+4)^2 = 0$$

$$y_c = C_1 e^{-4x} + C_2 x e^{-4x}$$

$$y(0) = C_1 = 0$$

$$y = C_2 x e^{-4x}$$

$$y' = C_2 e^{-4x} - 4C_2 x e^{-4x} \qquad y'(0) = C_2 = 12$$

$$\boxed{y = 12xe^{-4x}}$$

3. (15 pts) $y'' + 2y' + 5y = 60e^{3x}$

$$r^2 + 2r + 5 = 0$$

$$r = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$y_c = e^{-x}[c_1 \cos 2x + c_2 \sin 2x]$$

$$y_p = Ae^{3x}$$

$$y_p' = 3Ae^{3x}$$

$$y_p'' = 9Ae^{3x}$$

$$9Ae^{3x} + 6Ae^{3x} + 5Ae^{3x} = 60e^{3x}$$

$$20Ae^{3x} = 60e^{3x}$$

$$A = 3$$

$$y_p = 3e^{3x}$$

$$y = e^{-x}[c_1 \cos 2x + c_2 \sin 2x] + 3e^{3x}$$

4 (18pts) $y'' + 5y' + 6y = f(x)$

$$r^2 + 5r + 6 = 0$$

$$(r+3)(r+2) = 0$$

$$y_c = C_1 e^{-3x} + C_2 e^{-2x}$$

a) $y_p = Axe^{-3x}$

b) $y_p = Ax^2 + Bx + C$

c) $y_p = A\cos 6x + B\sin 6x + Cxe^{-2x}$

d) $y_p = e^{-2x}\left[A\cos 6x + B\sin 6x\right]$

5. (12pts)

$$y = xe^{rx}$$

$$y' = e^{rx} + rxe^{rx} = e^{rx}(1+rx)$$

$$y'' = re^{rx}(1+rx) + e^{rx}r$$

$$= e^{rx}(2r + r^2x)$$

$$ay'' + by' + cy = e^{rx}\left[2ar + \underline{ar^2x} + b + \underline{brx} + \underline{cx}\right]$$

$$= e^{rx}\left[x(\underbrace{\cancel{ar^2 + br + c}}_{=0}) + (2ar + b)_{=0}\right] = 0 \checkmark$$

Characteristic

$r = -\frac{b}{2a}$ since repeated

6. (17pts) $mx'' + bx' + kx = 0$

$F = ma$

$7 = m(32)$

$m = \frac{7}{32}$

$b = 5$

$F = kx$

$7 = k\left(\frac{1}{4}\right)$

$k = 28$

a)

$$\boxed{\frac{7}{32}x'' + 5x' + 28x = 0 \qquad x(0) = -\frac{7}{12} \qquad x'(0) = 0}$$

b) $b^2 - 4mk$

$25 - 4\left(\frac{7}{32}\right)28 = 25 - \frac{7}{8}(28) = 25 - \frac{7(14)}{4} =$

$25 - \frac{7(7)}{2} > 0$

overdamping

7. (6 pts)

$$\boxed{a_n = \frac{(-1)^n (n+2)}{3^{n-1}}}$$

8. (10pts)

a) $a_n = \frac{(-2)^n(-2)}{3^n} = \left(\frac{-2}{3}\right)^n(-2) \rightarrow \boxed{0}$ as $n \to \infty$

$\boxed{\text{converges}}$

b) No, it is neither increasing nor decreasing