9.7 Variation of Parameters

General solution to $\vec{x}' = A\vec{x} + \vec{f}$ is

$\vec{x} = \vec{x}_c + \vec{x}_p$

where

$\vec{x}_c$ is the solution to $\vec{x}' = A\vec{x}$ and

$\vec{x}_p = \Phi \int \Phi^{-1} \vec{f}\, dt$

Examples from Advanced Engineering Mathematics by Zill & Cullen p594

$\boxed{\text{Ex 1} \quad \text{Find the general solution} \quad \vec{x}' = \begin{pmatrix} 1 & 8 \\ 1 & -1 \end{pmatrix} \vec{x} + \begin{pmatrix} 12 \\ 12 \end{pmatrix} t}$

* 1st find $\vec{x}_c$

$|A - rI| = 0$

$\begin{vmatrix} 1-r & 8 \\ 1 & -1-r \end{vmatrix} = (1-r)(-1-r) - 8 = 0$

$r^2 - 9 = 0$

$r = \pm 3 \qquad r_1 = 3, \; r_2 = -3$

$(A - 3I)\vec{u} = \vec{0}$

$\begin{pmatrix} -2 & 8 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad u_1 = 4u_2 \qquad \vec{u}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} s$

P2

$$(A-(-3)I)\vec{u}=\vec{0}$$

$$\begin{pmatrix}4 & 8\\ 1 & 2\end{pmatrix}\vec{u}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$u_1+2u_2=0 \qquad \vec{u}_2=\begin{pmatrix}-2\\1\end{pmatrix}s$$

$$u_1=-2u_2$$

$$\vec{X}_c=C_1e^{3t}\begin{pmatrix}4\\1\end{pmatrix}+C_2\begin{pmatrix}-2\\1\end{pmatrix}e^{-3t}$$

$$\Phi=\begin{pmatrix}4e^{3t} & -2e^{-3t}\\ e^{3t} & e^{-3t}\end{pmatrix}$$

$\underline{\text{Recall}}$: $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ $\quad A^{-1}=\frac{1}{\det A}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$ if $\det A\neq 0$

$$\Phi^{-1}=\frac{1}{4+2}\begin{pmatrix}e^{-3t} & 2e^{-3t}\\ -e^{3t} & 4e^{3t}\end{pmatrix}=\begin{pmatrix}\frac{1}{6}e^{-3t} & \frac{1}{3}e^{-3t}\\ -\frac{1}{6}e^{3t} & \frac{2}{3}e^{3t}\end{pmatrix}$$

$$\vec{X}_p=\Phi\int\Phi^{-1}\vec{f}\,dt$$

$$=\Phi\int\begin{pmatrix}\frac{1}{6}e^{-3t} & \frac{1}{3}e^{-3t}\\ \frac{-1}{6}e^{3t} & \frac{2}{3}e^{3t}\end{pmatrix}\begin{pmatrix}12t\\12t\end{pmatrix}dt$$

$$=\Phi\int\begin{pmatrix}2te^{-3t}+4te^{-3t}\\ -2te^{3t}+8te^{3t}\end{pmatrix}dt=\Phi\int\begin{pmatrix}6te^{-3t}\\6te^{3t}\end{pmatrix}dt$$

Integration By parts $\quad uv-\int v\,du$

$$=\Phi\begin{pmatrix}\frac{6t}{-3}e^{-3t}-\int\frac{6}{-3}e^{-3t}\,dt\\ \frac{6t}{3}e^{3t}-\int\frac{6}{3}e^{3t}\,dt\end{pmatrix}$$

$$=\begin{pmatrix}4e^{3t} & -2e^{-3t}\\ e^{3t} & e^{-3t}\end{pmatrix}\begin{pmatrix}-2te^{-3t}-\frac{2}{3}e^{-3t}\\ 2te^{3t}-\frac{2}{3}e^{3t}\end{pmatrix}=\begin{pmatrix}-8t-\frac{8}{3}-4t+\frac{4}{3}\\ -2t-\frac{2}{3}+2t-\frac{2}{3}\end{pmatrix}$$

$$\boxed{\vec{X}=C_1e^{3t}\begin{pmatrix}4\\1\end{pmatrix}+C_2\begin{pmatrix}-2\\1\end{pmatrix}e^{-3t}+\begin{pmatrix}-12t-\frac{4}{3}\\ -\frac{4}{3}\end{pmatrix}}$$

P3 Ex 2

$$\boxed{\vec{X}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\vec{X} + \begin{pmatrix} \sec t \\ 0 \end{pmatrix}}$$

$$|A - rI| = 0$$

$$\begin{vmatrix} -r & -1 \\ 1 & -r \end{vmatrix} = r^2 + 1 = 0 \qquad r = \pm i$$

complex

$$(A - iI)\vec{u} = \vec{0}$$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$u_1 = u_2 i \qquad \vec{u} = \begin{pmatrix} i \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{a}} + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{b}} i$$

$$\vec{X_c} = C_1 e^{0t}\left(\cos t\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sin t\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + C_2 e^{0t}\left(\cos t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$= c_1\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

$$\Phi = \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix}$$

$$\Phi^{-1} = \frac{1}{-\sin^2 t - \cos^2 t}\begin{pmatrix} \sin t & -\cos t \\ -\cos t & -\sin t \end{pmatrix} = -1\begin{pmatrix} \sin t & -\cos t \\ -\cos t & -\sin t \end{pmatrix}$$

$$= \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix}$$

$$\vec{X_p} = \Phi \int \Phi^{-1}\vec{f} = \Phi \int \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix}\begin{pmatrix} \sec t \\ 0 \end{pmatrix} dt$$

$$= \Phi \int \begin{pmatrix} -\sec t \sin t \\ \cos t \sec t \end{pmatrix} dt = \Phi \int \begin{pmatrix} -\frac{\sin t}{\cos t} \\ 1 \end{pmatrix} dt$$

P4

u-substitution

$u = \cos t$

$du = -\sin t\,dt$

$$\vec{X}_p = \Phi \int \begin{pmatrix} \frac{1}{u}\,du \\ 1\,dt \end{pmatrix} = \Phi \begin{pmatrix} \ln|\cos t| \\ t \end{pmatrix}$$

$$= \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} \begin{pmatrix} \ln|\cos t| \\ t \end{pmatrix}$$

$$= \begin{pmatrix} -\sin t \ln|\cos t| + t\cos t \\ \ln|\cos t| \cos t + t\sin t \end{pmatrix}$$

$$\boxed{\vec{X} = c_1 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} t + \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \ln|\cos t|}$$

Ex 3 $\boxed{\vec{X}' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \vec{X} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t}$

$|A - rI| = 0$

$$\begin{vmatrix} -r & 2 \\ -1 & 3-r \end{vmatrix} = -r(3-r) + 2 = 0$$

$-3r + r^2 + 2 = 0$

$r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0 \quad r_1 = 2 \quad r_2 = 1$

$(A - r_1 I)\vec{u} = \vec{0}$

$$\begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad u_1 = u_2 \quad \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} s$$

$(A - r_2 I)\vec{u} = \vec{0}$

$$\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{matrix} -u_1 + 2u_2 = 0 \\ u_1 = 2u_2 \end{matrix} \quad \vec{u}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} s$$

P5) $\vec{X}_c = C_1 e^{2t}\begin{pmatrix}1\\1\end{pmatrix} + C_2 e^{t}\begin{pmatrix}2\\1\end{pmatrix}$

$$\Phi = \begin{pmatrix} e^{2t} & 2e^{t} \\ e^{2t} & e^{t} \end{pmatrix}$$

$$\Phi^{-1} = \frac{1}{e^{3t}-2e^{3t}} \begin{pmatrix} e^{t} & -2e^{t} \\ -e^{2t} & e^{2t} \end{pmatrix} = \begin{pmatrix} -e^{-2t} & 2e^{-2t} \\ e^{-t} & -e^{-t} \end{pmatrix}$$

$$\vec{X}_p = \Phi \int \Phi^{-1} \vec{f}$$

$$= \Phi \int \begin{pmatrix} -e^{-2t} & 2e^{-2t} \\ e^{-t} & -e^{-t} \end{pmatrix} \begin{pmatrix} e^{t} \\ -e^{t} \end{pmatrix} dt$$

$$= \Phi \int \begin{pmatrix} -e^{-t} - 2e^{-t} \\ 1 + 1 \end{pmatrix} dt$$

$$= \Phi \int \begin{pmatrix} -3e^{-t} \\ 2 \end{pmatrix} dt$$

$$= \begin{pmatrix} e^{2t} & 2e^{t} \\ e^{2t} & e^{t} \end{pmatrix} \begin{pmatrix} 3e^{t} \\ 2t \end{pmatrix} = \begin{pmatrix} 3e^{3t} + 4te^{t} \\ 3e^{3t} + 2te^{t} \end{pmatrix}$$

$$\boxed{\vec{X} = C_1 e^{2t}\begin{pmatrix}1\\1\end{pmatrix} + C_2 e^{t}\begin{pmatrix}2\\1\end{pmatrix} + 3\begin{pmatrix}e^{3t}\\e^{3t}\end{pmatrix} + \begin{pmatrix}4e^{t}t\\2e^{t}t\end{pmatrix}}$$

Ex 4

$$\boxed{\frac{dx}{dt} = 3x - 3y + 4 \qquad \frac{dy}{dt} = 2x - 2y - 1}$$

p.6.

$$A = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} \qquad \vec{f} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix}' = A\begin{pmatrix} x \\ y \end{pmatrix} + \vec{f}$$

$$\begin{vmatrix} 3-r & -3 \\ 2 & -2-r \end{vmatrix} = (3-r)(-2-r) + 6 = 0$$

$$\cancel{-6} + 2r - 3r + r^2 + \cancel{6} = 0$$

$$r^2 - r = 0 \qquad r_1 = 0 \quad r_2 = 1$$

$$(A - 0I)\vec{u} = \vec{0}$$

$$\begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad u_1 = u_2 \qquad \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_s$$

$$(A - I)\vec{u} = \vec{0}$$

$$\begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad 2u_1 - 3u_2 = 0$$

$$2u_1 = 3u_2 \qquad \vec{u}_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix}_s$$

$$u_1 = \tfrac{3}{2}u_2$$

$$\vec{X}_c = c_1 e^{0t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\underline{X} = \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix} \qquad \underline{X}^{-1} = \frac{1}{2e^t - 3e^t}\begin{pmatrix} 2e^t & -3e^t \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 3 \\ +1e^{-t} & -e^{-t} \end{pmatrix}$$

$$\vec{X}_p = \underline{X}\int \underline{X}^{-1}\vec{f} = \underline{X}\int \begin{pmatrix} -2 & 3 \\ e^{-t} & -e^{-t} \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix} dt = \underline{X}\int \begin{pmatrix} -8 - 3 \\ 4e^{-t} + e^{-t} \end{pmatrix} dt$$

$$= \underline{X}\int \begin{pmatrix} -11 \\ 5e^{-t} \end{pmatrix} dt = \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix}\begin{pmatrix} -11t \\ -5e^{-t} \end{pmatrix} = \begin{pmatrix} -11t - 15 \\ -11t - 10 \end{pmatrix}$$

$$\boxed{\vec{X} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} -11t - 15 \\ -11t - 10 \end{pmatrix}}$$