

5.2 HW: worked solutions

p250-251: 13, 15, 19, 25, 31

13. Find the general solution where differentiation is with respect to t .

$$\frac{dx}{dt} = x - 4y$$

$$\frac{dy}{dt} = x + y$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{bmatrix} 1-r & -4 \\ 1 & 1-r \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1-r)^2 + 4 = 0$$

$$1 - 2r + r^2 + 4 = 0 \quad r^2 - 2r + 5 = 0$$

$$r = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm \frac{4i}{2} = 1 \pm 2i$$

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$$r_1 = 1 + 2i$$

$$\begin{pmatrix} 1 - (1 + 2i) & -4 \\ 1 & 1 - (1 + 2i) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2ix - 4y = 0$$

$$y = -\frac{1}{2}ix$$

$$\vec{u}_1 = \begin{bmatrix} -2 \\ i \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \left[e^t \cos 2t \begin{bmatrix} -2 \\ 0 \end{bmatrix} - e^t \sin 2t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] \\ + C_2 \left[e^t \sin 2t \begin{bmatrix} -2 \\ 0 \end{bmatrix} + e^t \cos 2t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$$

$$x(t) = -2C_1 e^t \cos 2t - 2C_2 e^t \sin 2t$$

$$y(t) = -C_1 e^t \sin 2t + C_2 e^t \cos 2t$$

$$15. \frac{dw}{dt} = 5w + 2z + 5t$$

$$\frac{dz}{dt} = 3w + 4z + 17t$$

$$\begin{pmatrix} w \\ z \end{pmatrix}' = \begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} + \begin{pmatrix} 5t \\ 17t \end{pmatrix}$$

$$\begin{vmatrix} 5-r & 2 \\ 3 & 4-r \end{vmatrix} = (5-r)(4-r) - 6 = 0$$

$$20 - 9r + r^2 - 6 = 0$$

$$r^2 - 9r + 14 = 0$$

$$(r-7)(r-2) = 0$$

$$r_1 = 7 \quad r_2 = 2$$

$$r_1: \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$w = z \quad \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_s$$

$$r_2: \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3w = -2z \quad \vec{u}_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}_s$$

$$\begin{pmatrix} w \\ z \end{pmatrix}_p = at + b$$

$$\begin{pmatrix} w \\ z \end{pmatrix}'_p = a$$

$$a = \begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix} (at + b) + \begin{pmatrix} 5t \\ 17t \end{pmatrix}$$

$$a = Aat + Ab + \begin{pmatrix} 5t \\ 17t \end{pmatrix}$$

$$Aa = -\begin{pmatrix} 5 \\ 17 \end{pmatrix} \quad \begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 5a_1 + 2a_2 \\ 3a_1 + 4a_2 \end{pmatrix} = \begin{pmatrix} -5 \\ -17 \end{pmatrix}$$

$$a = Ab$$

$$5a_1 + 2a_2 = -5$$

$$3a_1 + 4a_2 = -17$$

$$-7a_1 = -7 \quad a_1 = 1$$

$$5(1) + 2a_2 = -5$$

$$a_2 = -5$$

$$\begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{pmatrix} 5b_1 + 2b_2 \\ 3b_1 + 4b_2 \end{pmatrix}$$

$$5b_1 + 2b_2 = 1$$

$$3b_1 + 4b_2 = -5$$

$$-7b_1 = -7 \quad b_1 = 1$$

$$b_2 = -2$$

$$\text{General } \begin{pmatrix} w \\ z \end{pmatrix} = c_1 e^{7t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ -5 \end{bmatrix} t + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$w(t) = c_1 e^{7t} + 2c_2 e^{2t} + t + 1$$

$$z(t) = c_1 e^{7t} - 3c_2 e^{2t} - 5t - 2$$

Solve the IVP

$$19. \frac{dx}{dt} = 4x + y \quad x(0) = 1$$

$$\frac{dy}{dt} = -2x + y \quad y(0) = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{vmatrix} 4-r & 1 \\ -2 & 1-r \end{vmatrix} = (4-r)(1-r) + 2 = 0$$

$$4 - 5r + r^2 + 2 = 0$$

$$r^2 - 5r + 6 = 0$$

$$(r-3)(r-2) = 0$$

$$r_1 = 3 \quad r_2 = 2$$

$$r_1: \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x = -y \quad \vec{u}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} s$$

$$r_2: \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} 2x &= -y \\ y &= -2x \end{aligned} \quad \vec{u}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} s$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2t}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$-C_1 + C_2 = 1$$

$$-C_2 = 1 \quad C_2 = -1$$

$$C_1 - 2C_2 = 0$$

$$C_1 = 2C_2$$

$$C_1 = -2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = -2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} - \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2t}$$

$$\boxed{\begin{aligned} x(t) &= 2e^{3t} - e^{2t} \\ y(t) &= -2e^{3t} + 2e^{2t} \end{aligned}}$$

$$25. \quad x' = x + 2y - z$$

$$y' = x + z$$

$$z' = 4x - 4y + 5z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{vmatrix} 1-r & 2 & -1 \\ 1 & -r & 1 \\ 4 & -4 & 5-r \end{vmatrix} = (1-r) \begin{vmatrix} -r & 1 \\ -4 & 5-r \end{vmatrix} - 2 \begin{vmatrix} 1 & 1 \\ 4 & 5-r \end{vmatrix} - \begin{vmatrix} 1 & -r \\ 4 & -4 \end{vmatrix}$$

$$= (1-r)[-5r + r^2 + 4] - 2[5-r-4] - [-4+4r]$$

$$= -5r + r^2 + 4 + 5r^2 - r^3 - 4r(-2) + 2r + 4 - 4r = 0$$

$$-r^3 + 6r^2 - 11r + 6 = 0$$

$$r^3 - 6r^2 + 11r - 6 = 0$$

$$r=1$$

$$(r-1)(r^2 + ar + b) = 0$$

$$\cancel{r^3} + ar^2 + br - r^2 - ar - b = r^3 - 6r^2 + 11r - 6$$

$$b=6$$

$$a-1 = -6$$

$$a = -5$$

$$(r-1)(r^2 - 5r + 6)$$

$$= (r-1)(r-3)(r-2)$$

$$r_1 = 1, r_2 = 3, r_3 = 2$$

$$r_1: \begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 4 & -4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2y = z$$

$$x - y + z = 0$$

$$\vec{u}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} s$$

$$x - y + 2y = 0$$

$$x = -y$$

$$r_2: \begin{pmatrix} -2 & 2 & -1 \\ 1 & -3 & 1 \\ 4 & -4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -4 & 1 \\ 1 & -3 & 1 \\ 4 & -4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$z = 4y$$

$$x - 3y + z = 0$$

$$x - 3y + 4y = 0$$

$$x = -y$$

$$\vec{u}_2 = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} s$$

$$r_3: \begin{pmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \\ 4 & -4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \\ 0 & 4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x + 2y - z = 0$$

$$4y = z$$

$$-x + 2y - 4y = 0$$

$$x = -2y$$

$$\vec{u}_3 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} s$$

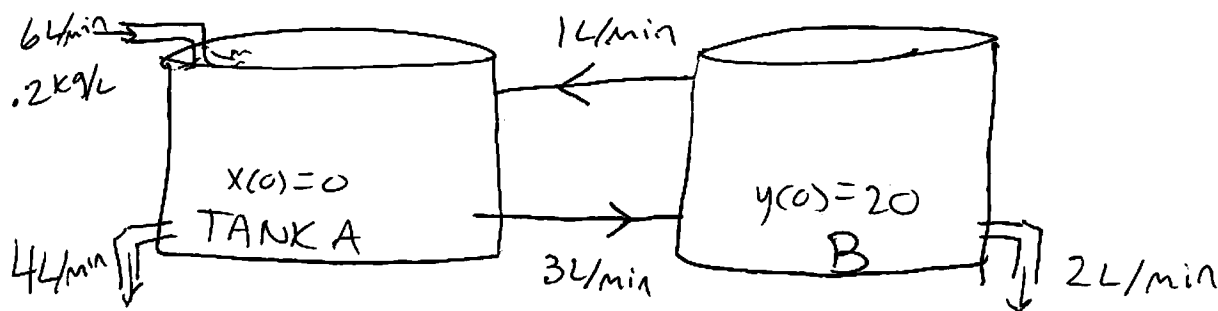
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = C_1 e^t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} + C_3 e^{2t} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

$$x = -C_1 e^t - C_2 e^{3t} - 2C_3 e^{2t}$$

$$y = C_1 e^t + C_2 e^{3t} + C_3 e^{2t}$$

$$z = 2C_1 e^t + 4C_2 e^{3t} + 4C_3 e^{2t}$$

31. 2 large tanks each hold 100 L of liquid & are interconnected by pipes with the liquid flowing from tank A into tank B at a rate of 3 L/min & from B into A at a rate of 1 L/min. The liquid is kept well-stirred. A brine solution with a concentration of 0.2 kg/L of salt flows into tank A at a rate of 6 L/min. The solution flows out of the system from tank A at 4 L/min & from tank B at 2 L/min. If, initially, tank A contains pure water & tank B contains 20 kg of salt, determine the mass of salt in each tank at time t .



* Notice the volume in the tanks is not changing

$$x' = \text{Rate in} - \text{Rate out}$$

$$y' = \text{Rate in} - \text{Rate out}$$

$$x' = (0.2 \text{ kg/L}) (6 \frac{\text{L}}{\text{min}}) + \left(\frac{y}{100}\right) (1 \frac{\text{L}}{\text{min}}) - \frac{x}{100} (4 \frac{\text{L}}{\text{min}}) - \frac{x}{100} (3 \frac{\text{L}}{\text{min}})$$

$$x' = 1.2 + \frac{y}{100} - \frac{7x}{100}$$

$$y' = \frac{x}{100} (3 \frac{\text{L}}{\text{min}}) - \frac{y}{100} (1 \frac{\text{L}}{\text{min}}) - \frac{y}{100} (2 \frac{\text{L}}{\text{min}})$$

$$= \frac{3x}{100} - \frac{3y}{100}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -\frac{7}{100} & \frac{1}{100} \\ \frac{3}{100} & -\frac{3}{100} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1.2 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} -\frac{7}{100} - r & \frac{1}{100} \\ \frac{3}{100} & -\frac{3}{100} - r \end{vmatrix} = \left(-\frac{7}{100} - r\right) \left(-\frac{3}{100} - r\right) - \frac{3}{100^2}$$

$$r^2 + \frac{10r}{100} + \frac{18}{100^2}$$

$$r = \frac{-\frac{10}{100} \pm \sqrt{\frac{100}{100^2} - \frac{72}{100^2}}}{2} = \frac{-\frac{10}{100} \pm \frac{\sqrt{28}}{100}}{2} = \text{sadly these are the correct values of } r.$$

For a test:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -\frac{7}{100} & \frac{1}{100} \\ \frac{3}{100} & -\frac{3}{100} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1.2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 20 \end{pmatrix}$$