12.3. Worksheet

Setting up double integrals, reversing the order of integration, finding volumes

Setting up Double Integrals:

p900

19. \( \iint \frac{\partial y}{\partial x} \, dA \), where \( D \) is the region in the 1st quadrant bounded by the parabolas \( x = y^2 \) and \( x = 8 - y^2 \)

* Start these problems by drawing a picture

\[
\begin{align*}
\text{y} & \\
\text{x} & \\
\text{x = y}^2 & \\
\text{x = 8 - y}^2
\end{align*}
\]

* This is what your book refers to as a Type II region (p845). Basically, \( D \) is between \( x = h_1(y) \) and \( x = h_2(y) \). For these problems, draw an arrow from the left curve to the right curve. This gives us our \( x \) bounds, \( y^2 \leq x \leq 8 - y^2 \)
To find $y$ just find the points of intersection of the curves.

$8 - y^2 = y^2$

$8 = 2y^2$

$4 = y^2$

$y = \pm 2$

Not in the 1st quadrant

$$SSD = \int_0^2 \int_{y^2}^{8-y^2} y \, dx \, dy$$

$$= \int_0^2 y \left[ x \right]_{y^2}^{8-y^2} \, dy$$

$$= \int_0^2 8y - y^3 - y^3 \, dy$$

$$= \left[ 4y^2 - \frac{2}{4} y^4 \right]_0^1 = 16 - 8 = 8$$

$$SSD \cdot \frac{1}{1 + x^2} \, dA$$

where $D$ is bounded by $y = \sqrt{x}, y = 0, x = 1$

* First draw $D$

* This is a Type 1 region and a Type 2 region

* First we'll do it as a Type 1 ($D$ is bounded by $y = h_1(x)$ and $y = h_2(x)$)

Draw an arrow from the bottom curve to the top curve. These are your $y$ bounds

$0 \leq y \leq \sqrt{x}$
To find \( x \) remember that the outermost limits of integration must be constants. Just find the smallest & largest \( x \) can be.

\[
\int_0^1 \int_0^{\frac{1}{\sqrt{x}}} \frac{u}{1+x^2} \, dy \, dx = \int_0^1 \frac{\frac{1}{\sqrt{x}}}{1+x^2} \, du
\]

\[
= \int_0^1 \frac{1}{x} \cdot \frac{1}{1+x^2} \, dx \quad u=1+x^2 \quad du=2x \, dx
\]

\[
u(0)=1+0^2=1 \quad u(1)=1+1^2=2
\]

\[
= \int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u} \, du = \frac{1}{4} \ln|u| \bigg|_1^{\frac{1}{\sqrt{2}}} = \frac{1}{4} \ln 2 - \frac{1}{4} \ln 1 = \frac{1}{4} \ln 2
\]

Alternatively, you could have done this problem as a type 2 region (it is harder this way).

\[
\begin{array}{c}
y = \sqrt{x} \\
\end{array}
\]

* Draw an arrow going from the left curve to the right curve & solve for \( x \) if necessary to get

\[
y^2 \leq x \leq 1
\]

\( y \) goes from 0 to \( \sqrt{1} = 1 \)

\[
\int_0^1 \int_0^{\sqrt{x}} \frac{y}{1+x^2} \, dx \, dy
\]
Reversing the Order of Integration

p850

* To change the order of integration, just draw \( D \), the integrand doesn't matter!

34. \( \int_4^1 \int_{4x}^{4} f(x,y) \, dy \, dx \)

This tells us that \( 0 \leq x \leq 1 \) and \( 4x \leq y \leq 4 \)

Since I want to have \( \, dx \, dy \), I know \( x \) is the inside function. Draw an arrow from the left curve to the right curve & solve for \( x \) if necessary

\[ 0 \leq x \leq \frac{y}{4} \]

\[ 0 \leq y \leq 4 \quad \text{biggest} \]
\[ y \leq 4 \quad \text{smallest} \]

\[ 6 \int_4^1 \int_{4x}^{4} f(x,y) \, dy \, dx = \int_0^4 \int_0^{y/4} f(x,y) \, dx \, dy \]
36. \[ \int_{0}^{3} \int_{0}^{\sqrt{9-y^2}} f(x,y) \, dx \, dy \]

- \[0 \leq y \leq 3\]
- \[0 \leq x \leq \sqrt{9-y^2}\]

* Draw \( D \)

* Draw an arrow from the bottom curve to the top curve and solve for \( y \) if necessary.

\[ 0 \leq y \leq \sqrt{9-x^2} \]
\[ 0 \leq x \leq 3 \]

\[ \int_{0}^{3} \int_{0}^{\sqrt{9-x^2}} f(x,y) \, dy \, dx \]
Volumes in Double Integrals

The volume of the solid that lies below the surface given by \( z = f(x, y) \) and above the region \( D \) in the \( xy \)-plane is

\[
V = \iint_D f(x, y) \, dA
\]

p900 Find the volume of the given solid

29. Under the paraboloid \( z = x^2 + 4y^2 \) and above the rectangle \( R = [0, 2] \times [1, 4] \)

* Draw a picture

R = [0, 2] \times [1, 4]
tells us
\[ 0 \leq x \leq 2 \]
\[ 1 \leq y \leq 4 \]

\[
\int_0^2 \int_1^4 x^2 + 4y^2 \, dy \, dx \quad \text{or} \quad \int_1^4 \int_0^2 x^2 + 4y^2 \, dx \, dy
\]

To evaluate this using a triple integral:

\[
\int_0^2 \int_1^4 \int_0^2 1 \, dz \, dy \, dx \quad \text{or} \quad \int_1^4 \int_0^2 \int_0^2 1 \, dz \, dx \, dy
\]

You can finish integrating these on your own.
30. Find the volume under the surface $z = x^2y$ and above the triangle in the $xy$-plane with vertices $(1,0)$, $(2,1)$ and $(4,0)$.

* We already know $f(x,y) = x^2y$ is our integrand so let's just figure out what is happening in the $xy$-plane.

* This is another Type 2 region; we draw an arrow from the left curve to the right curve.

So there are 2 lines we need to find equations for and then solve for $x$.

$$m_1 = \frac{\Delta y}{\Delta x} = \frac{1-0}{2-1} = 1$$

$$y - y_1 = m(x - x_1) \quad y - 0 = 1(x - 1) \quad y = x - 1 \quad \text{is } L_1$$

$$m_2 = \frac{\Delta y}{\Delta x} = \frac{0-1}{4-2} = -\frac{1}{2}$$

$$y - y_1 = m(x - x_1) \quad y - 0 = -\frac{1}{2}(x - 4) \quad y = -\frac{1}{2}x + 2 \quad \text{is } L_2$$

* Solve $L_1$ and $L_2$ for $x$ to get

$$x = y + 1 \quad \text{and} \quad x = -2y + 4 \quad \text{L_1 and L_2}$$
\[ y+1 \leq x \leq 4-2y \]
\[ 0 \leq y \leq 1 \]
\[ \int_0^1 \int_{y+1}^{4-2y} x^2 y \, dx \, dy \] (This is a bit messy when it comes to integration)