

# A compact set of compactified moduli spaces

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November 10, 2006

Version of November 10, 2006

## **Abstract**

A survey of various moduli spaces, their various compactifications and uses

The most classical moduli space is the moduli space  $\mathcal{M}_g$  of genus  $g$  Riemann surfaces.

Topologically there is just one underlying surface for each genus  $g$ . Two Riemann surfaces of genus  $g$  are considered equivalent if there is a complex analytic homeomorphism between them with complex analytic inverse - biholomorphic equivalence if you prefer.

For genus 0, there is only one equivalence class. For  $g > 0$ , the equivalence classes are well described as a complex algebraic manifold of complex dim  $3g-3$ .

To a topologist, that this moduli space has the same kind of structure as the members of the family again the same structure attracts little notice, but, for algebraic geometers it is very attractive; unfortunately, it sometimes fails to happen, leading to things like stacks.

Over the years, the term moduli space has been applied more generally to equivalence classes of structures. A prominent example is

The moduli space  $\mathcal{M}_n(X)$  of configurations of  $n$ -tuples of distinct points in a background space  $X$ , up to a class of homeomorphisms of the background.

Consider the moduli space of configurations on the real line up to affine equivalence. Affine equivalence allows us to consider the first and last points to be 0 and 1, so the moduli space is homeomorphic to the interior of the  $n - 2$ -simplex.

$$0 \text{-----} x_1 \text{-----} x_2 \cdots x_{n-2} \text{-----} 1$$

It would be trivial to compactify this as the closed simplex, but an alternative is to compactify so as to reveal what is happening as points approach collision. When three points come together simultaneously, we can still distinguish cases according to the relative speed of approach to collision.

We can compactify so as to reveal this information. See illustrations in

Notices of the AMS, June/July 2004

or

math-ph/0405011 [abs, ps, pdf, other] :

Combinatorial equivalence of real moduli spaces

by Satyan L. Devadoss

This is one way of arriving at the associahedra, decades after they were first constructed.

If the background is the circle  $S^1$ , an appealing picture is that of a bubble tree. In Kontsevich's terms, this can be regarded as obtained by successive magnification.

Another important background is a complex manifold of genus  $g$ . For the case  $g = 0$ , i.e. a Riemann sphere, the Deligne-Knudsen-Mumford compactification of the moduli space

$$\mathcal{M}_{0,p} := ((\mathbb{CP}^1)^n \setminus \Delta) / \mathrm{PGL}(2, \mathbb{C})$$

of  $p$ -punctured complex projective lines  $\mathbb{CP}^1$ .

Here

$$\begin{aligned} \Delta = \{\text{diagonals}\} = \{ & (z_1, \dots, z_p) \in (\mathbb{CP}^1)^n \\ & z_i = z_j \text{ for some } i \neq j\}. \end{aligned}$$

The compactification  $\overline{\mathcal{M}}_{0,p}$  is itself the moduli space of stable  $p$ -punctured complex curves, which include nonsingular punctured projective lines as well as degenerations of them of a certain kind.

A *stable  $p$ -punctured complex curve of genus 0* is a connected compact complex curve  $C$  of genus 0 with  $p$  punctures, such that (1) it may have ordinary double points away from the punctures, (2) each irreducible component of the curve  $C$  is a projective line and (3) the total number of punctures and double points on each component of  $C$  is at least 3. Both  $\mathcal{M}_{0,p}$  and  $\overline{\mathcal{M}}_{0,p}$  are complex algebraic manifolds of complex dimension  $p - 3$ . The moduli space  $\mathcal{M}_{0,p}$  of nonsingular curves is an open submanifold in the projective manifold  $\overline{\mathcal{M}}_{0,p}$ .

One can visualize the degeneration of a punctured projective line (a punctured Riemann sphere) as a process where the sphere undergoes “mitosis” into two spheres by forming a long thin neck away from the punctures; the points which are headed for collision are instead in the sphere which bubbles off from the original sphere.. In the limit, the sphere degenerates into two spheres joined at a double point. The degenerations must all be stable, that is, there must be at least three punctures or double points on each irreducible component.

For general  $g$ , the analogous construction goes through and the results are sometimes denoted  $\mathcal{M}_{g,p}$  and  $\overline{\mathcal{M}}_{g,p}$ , but there is a variety (no pun intended) of others.

Of course one need not stop with Riemann surfaces without boundary. Here the appropriate generality can be denoted  $\overline{\mathcal{M}}_{g,b;p,q}$  to indicate  $p$  punctures in the interior (the bulk) and  $q$  punctures on the boundary.

One way to picture the compactification is by ‘blowing up’; that is, excising an open tubular neighborhood of the diagonal  $\Delta$  to get a manifold with boundary, in fact, with corners OR closing up the result by passing to the projective space of the spherical fibres.

Another way to analyze a moduli space is to add extra structure. This is the theme of Mumford’s GIT.

Another kind of extra structure is to decorate the punctures with local coordinates (Huang and Barron) vanishing at the punctures or by disjoint little disks centered at the punctures.

We can also consider unordered configurations of points. For example,  $\bar{M}_{0,n}(\mathcal{R})$ , the real points of the compactified moduli space of points on the sphere is tiled by associahedra, and can be constructed from iterated blow-ups of certain strata of the Coxeter complex of type A ( again see Devadoss).

The composition of stable curves described above provides the collection  $\overline{\mathcal{M}}_{p+1}$  with the structure of an operad of algebraic varieties. This structure is compatible with the fundamental operad structure on trees.

From the point of view of symplectic geometry, the quotient is the configuration space of  $n$ -gon linkages in  $\mathbb{R}^3$  modulo  $SU(2)$  (or  $SO(3, \mathbb{R})$ ) with prescribed side lengths.

Still in a topological or geometric context, one can multiply examples:

moduli spaces of rational maps

moduli space of  $G$ -bundles

etc.

The geometry can be differential or algebraic. Venturing into the algebraic, moduli spaces are intimately related to deformation theory. Indeed, deformations of an algebra or other algebraic structure are considered up to isomorphism; the set of such isomorphism classes is the related moduli ‘space’. An issue of importance is giving it a topology or homotopy type or algebraic structure itself of some sort. That latter can be problematical:

Vakil - Murphy's Law in algebraic geometry: Badly-behaved deformation spaces  
math.AG/0411469

He writes:

“We consider the question: ‘How bad can the deformation space of an object be?’ The answer seems to be: ‘Unless there is some a priori reason otherwise, the deformation space may be as bad as possible.’ We show this for a number of important moduli spaces.

The objects themselves are not pathological, and are in fact as nice as can be. This justifies Mumford's philosophy that even moduli spaces of well-behaved objects should be arbitrarily bad unless there is an a priori reason otherwise.”