Solving the A_∞-Case of Deligne's Conjecture with Polytopes

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Outline

- Basic Definitions and Notation
- Associahedra and Cyclohedra as Compactifications of Moduli Spaces
- A “New” Realization of Associahedra and Cyclohedra
- Two-and-a-Half CW complexes and Their Homotopy Equivalence
- Solving the A_∞-Deligne Conjecture

Basic Definitions and Notation

A **rooted, planted** tree τ is one with a marked vertex that lies on a unique edge (usually placed at bottom). The marked vertex is the **root**, and the unique edge adjacent to the root is the **root edge**.

Let E_τ and V_τ be the set of edges and vertices resp. of τ, E_v be the edges adjacent to v, and let An_v be the set of angles at the vertex v created by E_v.

If τ is oriented toward the root, $N(v)$ is the next vertex after v. The **valence** $|v|$ of a vertex v of τ is $|E_v|$, or $|N^{-1}(v)| + 1$.

A **leaf** is a vertex with $|N^{-1}(v)| = 0$. A **leaf edge** is an edge adjacent to a leaf, and an **internal edge** is a non-leaf, non-root edge. A **corolla** is a tree with no internal edges. Let E_{int} be the set of internal edges of τ.

A **pinning** is a cyclic ordering of the set of adjacent edges to each vertex.

Note: Pinning + rooted gives linear order on the vertices and edges.

A **planar** tree is a tree together with a pinning. It can be embedded in the plane so that the pinning matches the natural orientation of the plane (counterclockwise).

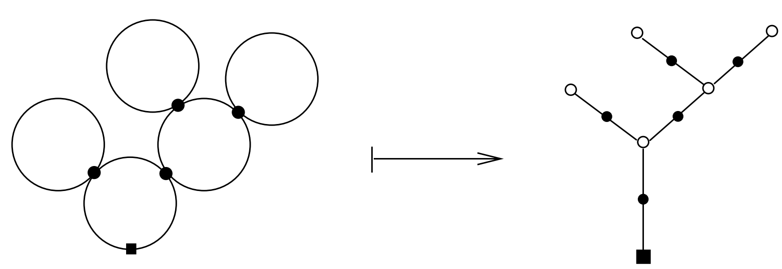
A **labelling** of a tree τ is a bijection from a finite set L to $\{$leaves$\}, \{$edges$\}$, etc. of τ.

A **coloring** of a tree τ is a bijection from $\{$black, white$\}$ to the vertices of τ. In such a tree, let $V_b(\tau)$ and $V_w(\tau)$ be the set of black and white vertices of τ resp.

A **stable** tree is one such that $|v_b| \geq 3$ for all black vertices v_b. Let $\mathcal{T}_\infty$ denote the vectorspace generated by the stable, planar, planted trees.

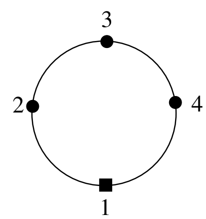
A **bipartite tree** is one such that all edges only join black vertices to white vertices. Let $\mathcal{T}_{bp}$ denote the vectorspace generated by the bipartite, planar, planted trees.

A **cactus** is a tree-like configuration of S^1's with one base point and points of intersection between circles marked. Let $Cact^1$ be the operad of cacti. Then each element $c \in Cact^1$ can be identified with a tree $\tau_c \in T_{bp}$ by shrinking each S^1 to a white vertex and marking each point of intersection by a black vertex.



Associahedra and Cyclohedra as Compactifications of Moduli Spaces

View a moduli space as n moving points on a circle with one fixed point; points can't collide:

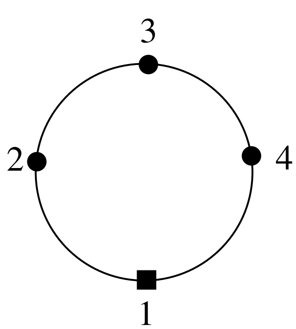


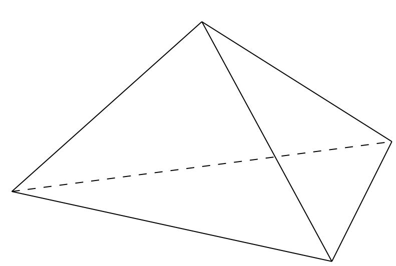
Order of collision doesn't matter; all points can collide:

- $\triangle^n$ ($(n-1)$-dimensional simplex)
- k collisions in any order $\leftrightarrow$ $(n-k-1)$-dimensional face.

E.g. Two collisions give the six edges (dim 4-2-1=1) of $\triangle^4$, given by:

(123)4	12)3(4
1(234)	(12)(34)
1)2(34	1)(23)(4





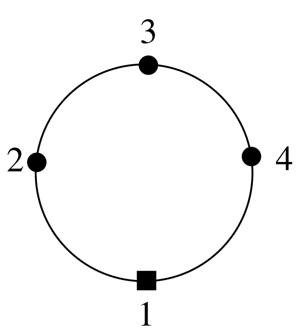
$\mathbf{K_n}$

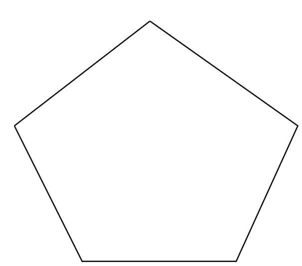
Order matters; can't collide with the marked point:

- K_{n-1} (associahedron of dimension $n-3$)
- k bracketings $\leftrightarrow$ ($n-k-3$)-dimensional face).

<u>E.g.</u> Two bracketings give the five vertices (dim 5-2-3=0) of K_4, given by:

((12)3)4 (1(23))4
1((23)4) 1(2(34))
(12)(34)





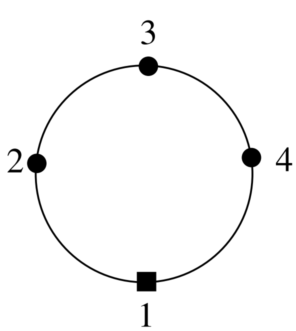
$\mathbf{W_n}$

Order matters; can collide with marked point:

- W_n (cyclohedron of degree $n-1$)
- k bracketings $\leftrightarrow$ $(n-k-1)$-dimensional face).

E.g. Two bracketings give the 29 edges of W_4, given by:

(1(23)4)
1)((23)4, etc.



A "New" Realization of K_n and W_n (as CW-complexes)

- $\mathcal{T}^b_\infty(n)$:=$\{\tau \in \mathcal{T}_\infty$: has only black vertices and n leaves
- Consider (τ, H) and (τ, H^{cell}) for $\tau \in \mathcal{T}^b_\infty(n)$

$H : E_{int} \to [0,1] = (h^\tau_i)^{n-2}_{i=1}$ (τ has a linear order on the edges)

$H^{cell} : E_{int} \to \{0, 1, x\}$

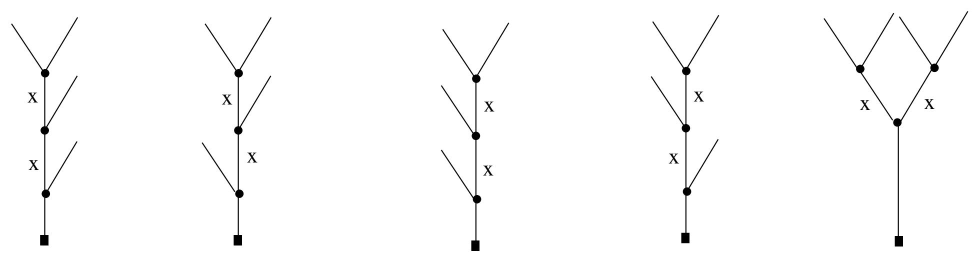
- If $H(e) = 0$ for $e \in E_\tau$, then $\tau \sim \tau/e$

Prop: K_n is the following space/complex:

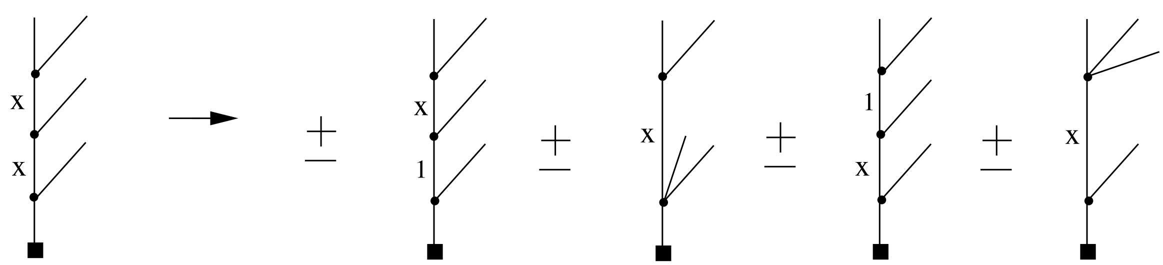
- Points are (τ, H), for $\tau \in \mathcal{T}^b_\infty(n)$
- Cells indexed by (τ, H^{cell}), for $\tau \in \mathcal{T}^b_\infty(n)$; call the cell $c(\tau, H^{cell})$
- Dim$(c(\tau, H^{cell})) = |(H^{cell})^{-1}(x)|$
- Glue: the boundary of $c(\tau, H^{cell})$ is given by $\sum \pm$ (all cells whose trees represent sending x's to 0 or 1 one at a time (if they weren't already)). Call this map ∂.

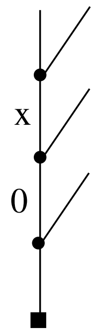
E.g. K_4

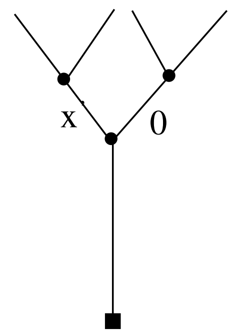
There are five 2-cells, given by the five binary trees with 2 interior edges and 4 leaves with all internal edges assigned x:

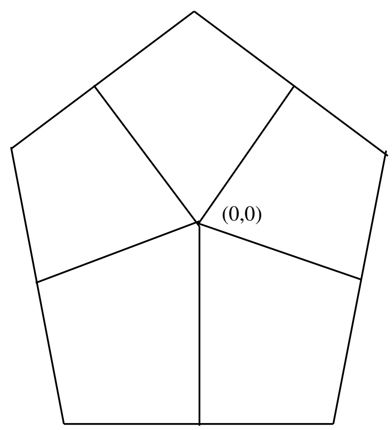


τ_1, for example, will have boundary:









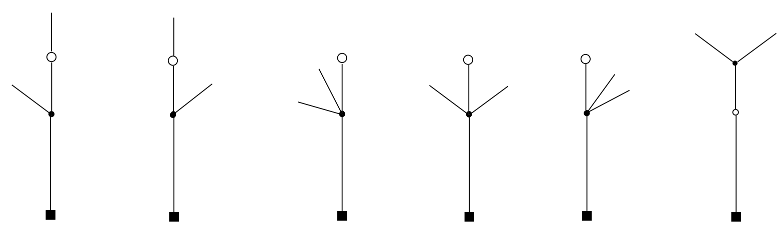
- $K_n = N$ copies of $I^{|(H^{cell})^{-1}(x)|} = N$ copies of $I^{(n-2)}$ glued to one other, one along each $(n-3)$-dim face (known from Boardmann-Vogt)
- $N = |\{\text{binary trees in } \mathcal{T}_\infty^b(n)\}|$

$\underline{W_n}$**:** index with a white $(n-1)$-corolla

- Points of W_n are all the trees that can be obtained from adding black vertices and edges to the white $(n-1)$ corolla in a stable way + height set $H : E_{int} \to [0,1]$ + angle set A:$An_{v_w} \to [0,1] = (t_i)$ such that $\sum t_i = 1$.

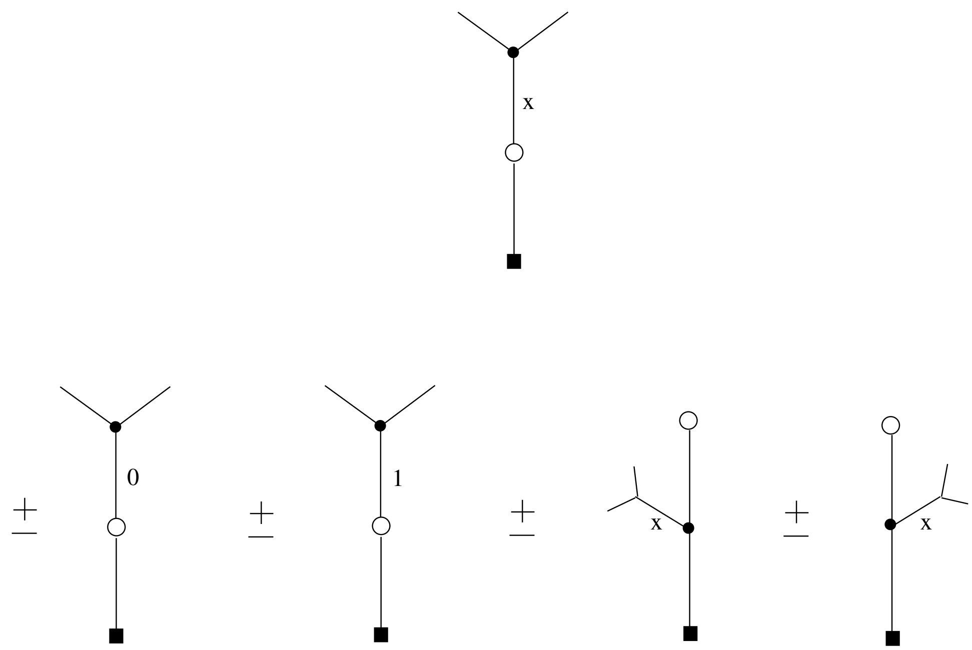
- Cells of W_n indexed by trees from above + height function $H^{cell} : E_{int} \to \{0,1,x\}$; call the cell $c(\tau, H)$.

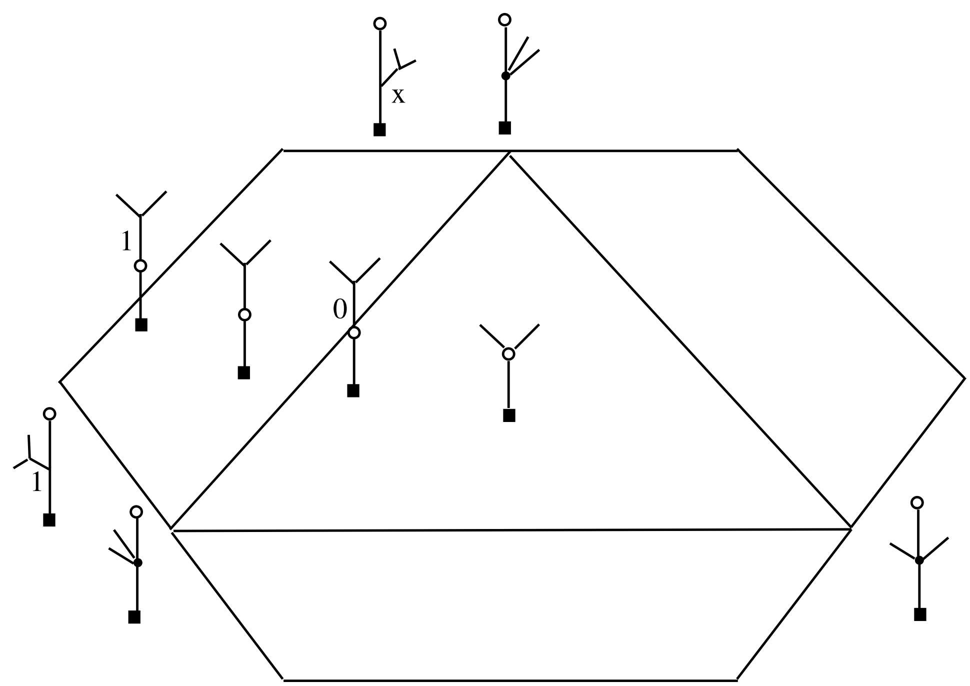
- Do not contract edges assigned 0 unless they are joined by two black vertices



- Cells have dimension $d = |(H^{cell})^{-1}(x)| + |v_w| - 1$

- Glue: the boundary components of $c(\tau, H^{cell})$ are given by $\sum \pm$ (all cells whose trees represent sending x's to 0 or 1 (if they weren't already)) + $\sum \pm$ (all cells whose trees represent having collapsed an angle at a white vertex). Call this map ∂. For example:


x
±
0
±
1
±
x
±
x


x
1
0
1

Two-and-a-Half CW Complexes

$\underline{X_\infty}$:

- Cells indexed by:

$C(\tau) = \left(\prod_{v \in V_b(\tau)} K_{|N^{-1}(v)|}\right) \times \left(\prod_{v \in V_w(\tau)} W_{|v|}\right)$, for each $\tau \in \mathcal{T}_\infty$

- Gluing map ∂_∞ given by $\partial_\infty(\tau) = \sum_{v \in V_\tau} \sum$ (all trees obtained by adding a black vertex and an edge at the vertex v maintaining stability)

$\underline{X^H_\infty}$:

- Extend H to $\mathcal{T}_\infty$ by H(white edge)$= 0$
- Cells indexed by:

$C(\tau, H^{cell}) = \left(\prod_{v \in V_w(\tau)} \triangle^{|v_w|}\right) \times \left(\prod I^{|(H^{cell})^{-1}(x)|}\right)$, for each $\tau \in \mathcal{T}_\infty$ with height function H^{cell}

- Gluing map ∂^H_∞ is simply ∂ on (products of) simplices

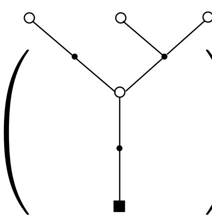
$|X_\infty| = |X^H_\infty|$

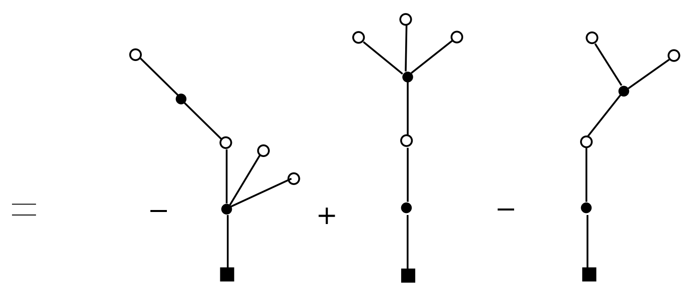
$\underline{X_1}$:

- Cells indexed by:

$\triangle(\tau) = \prod_{v \in V_w(\tau)} \triangle^{|v|}$, for each $\tau \in \mathcal{T}_{bp}$

- $\triangle(\tau_1 \pm \tau_2) = \triangle(\tau_1) \pm \triangle(\tau_2)$
- Gluing map $= \partial_{bp}$:

$\partial_{bp}\Bigg($  $\Bigg) =$



Prop: $h : X_\infty^H \to X_1$ is a homotopy equivalence, where h is the map that contracts all K_n to a point by letting $h_i^\tau \to 0 \;\forall\; h_i^\tau$.

Deligne's Conjecture

If A is an A_∞ algebra (acted upon by the K_n operad), then there is a chain model of the little disks operad D_2 that acts on $CH^n(A, A)$, the Hochschild cohomology of A.

<u>Proof:</u>
(1) $CC_*(X_\infty)$ is a chain model for D_2
(2) $CC_*(X_\infty) \cong M$ by construction.
(3) If A is A_∞, then $CH^n(A, A)$ form an algebra over M. (Kontsevich-Soibelman)

$$X_\infty^H \xrightarrow{h_\infty} X_1 \overset{\text{Kauf}}{\approx} D_2$$

$$CC_*(X_\infty) \overset{quis}{\hookrightarrow} CC_*(X_1)$$

$$H_*(X_\infty) \xrightarrow{H_*(h_\infty)} H_*(X_1) \overset{\text{Kauf}}{\cong} H_*(D_2)$$

References

- Ralph M. Kaufmann, *Operads, moduli of surfaces and quantum algebras.* Woods Hole mathematics, 133–224, Ser. Knots Everything, 34, World Sci. Publ., Hackensack, NJ, 2004.

- Martin Markl, Steve Shnider, and Jim Stasheff, **Operads in Algebra, Topology, and Physics**, Mathematical Surveys and Monographs, Vol.96, AMS, Providence, RI, (2002)

- Bott, Raoul; Taubes, Clifford *On the self-linking of knots. Topology and physics.* J. Math. Phys. 35 (1994), no. 10, 5247–5287.

- M. Kontsevich and Y. Soibelman. *Deformations of algebras over operads and Deligne's conjecture.* Conférence Moshé Flato 1999, Vol. I (Dijon), 255-307, Math. Phys. Stud., 21, Kluwer Acad. Publ., Dordrecht, 2000.

- R. M. Kaufmann. *On several varieties of cacti and their relations.* Algebraic & Geometric Topology 5 (2005), 237-300.

- R. M. Kaufmann. *On Spineless Cacti, Deligne's Conjecture and Connes-Kreimer's Hopf Algebra.* Topology, to appear.

- Satyan L. Devadoss, *A space of cyclohedra.* Discrete & Computational Geometry 29 (2003) 61-75