N-differential graded algebras: examples and applications

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The ideas presented in this talk are developed in the papers:


Introduction

The goal of this talk is to introduce the notion of

\( N \)-differential graded algebras.

According to Kapranov and Mayer a \( N \)-complex over a field \( k \) is a

\( \mathbb{Z} \)-graded \( k \)-vector space

\[ V = \bigoplus_{n \in \mathbb{Z}} V_n \]

together with a degree one linear map

\[ d : V \to V \text{ such that } d^N = 0. \]
There are two generalizations of the notion of differential graded algebras to the context of $N$-complexes.

A choice, introduced by Kerner and further studied by Dubois-Violette and Kapranov, begins by fixing a primitive $N$-th root of unity $q$.

A $q$-differential graded algebra $A$ is a $\mathbb{Z}$-graded associative algebra together with a degree one linear operator

$$d : A \rightarrow A$$

such that

$$d(ab) = d(a)b + q^{\bar{a}}ad(b) \quad \text{and} \quad d^N = 0.$$

There are many interesting examples and constructions of $q$-differential graded algebras.
In this talk we consider another choice, the notion of $N$-differential graded algebras.

A $N$-differential graded algebra ($N$-dga) over a field $k$ is a triple $(A, m, d)$ where $m : A \otimes A \to A$ and $d : A \to A$ are linear maps such that:

1. $d^N = 0$, that is $(A, d)$ is a $N$-complex.
2. $(A, m)$ is a graded associative algebra.
3. $d$ satisfies the graded Leibnitz rule $d(ab) = d(a)b + (-1)^\bar{a}ad(b)$. 
Categorical justification of the notion of $N$-dga.:

Let Nil-dg vect be the category of nilpotent complexes.

A nilpotent differential graded vector spaces is a pair $(V, d)$ such that $(V, d)$ is a $N$-complex for some integer $N \geq 1$.

Similarly we define the category Nil-dga of nilpotent differential graded algebras.

Objects in Nil-dga are pairs $(A, d)$ such that $(A, d)$ is a $N$-dga for some integer $N \geq 1$. 
We have the following result:

• The category Nil-dgvect is a symmetric monoidal category.

   Indeed if \((V, d_V)\) is a \(N\)-complex and \((W, d_W)\) is a \(M\)-complex then

   \[
   (V \otimes W, d_V \otimes Id + Id \otimes d_W)
   \]

   is a \(N + M - 1\)-complex.

• Nil-dga is the category of monoids in Nil-dgvect.

• Nil-dga inherits a symmetric monoidal structure from Nil-dgvect.
In the rest of the talk I want to address the following issues:

- Examples of $N$-differential graded algebras.
- Deformations of $N$-differential graded algebras.
- $N$ Lie algebroids.
- $A_{\infty}^N$-algebras.
Examples of $N$-differential graded algebras.

The simplest construction of $N$-dga: $N$-flat connections.

Let $M \times V$ be a trivial vector bundle over $M$.

A connection

$$\omega \in \Omega^1(M, \text{End}(V)) \text{ on } M \times V$$

is an

$\text{End}(V)$-valued 1-form on $M$.

The connection $\omega$ defines a covariant derivative

$$d + \omega : \Omega(M, V) \to \Omega(M, V)$$
Proposition: \( (\Omega(M, V), d + \omega) \) is a \( 2N \)-differential graded algebra if and only if \( \omega \) is a \( N \)-flat connection, that is the curvature

\[
F = d\omega + \omega \wedge \omega
\]

of \( \omega \) satisfies the identity

\[
F^N = 0.
\]

Example. Consider \( \mathbb{R}^4 \) with coordinates \((x_1, x_2, x_3, x_4)\).

Given smooth functions \( \omega_1, \omega_2 : \mathbb{R}^2 \to \mathbb{R} \) consider the connection

\[
\omega = \omega_1 dx_1 + \omega_2 dx_2.
\]

A simple calculation shows that

\[
F = \left( \frac{\partial \omega_1}{\partial x_2} - \frac{\partial \omega_2}{\partial x_1} \right) dx_1 \wedge dx_2 \neq 0, \quad \text{if} \quad \frac{\partial \omega_1}{\partial x_2} - \frac{\partial \omega_2}{\partial x_1} \neq 0.
\]
For example one can take

\[ \omega_1 = x_2 \quad \text{and} \quad \omega_2 = -x_1, \]

so we have

\[ F = 2dx_1 \wedge dx_2 \neq 0. \]

Moreover it is clear that

\[ F^2 = 0. \]

Thus \( \omega \) is a 2-flat connection and therefore

\[ (\Omega(\mathbb{R}^4), d + \omega) \text{ is a 4-dga.} \]
Another geometric source of $2N$-differential graded algebras are certain kind of Riemannian manifolds.

Let $(M, g)$ be a Riemannian manifold with associated covariant derivative $\nabla$.

Assume that the tangent space $TM$ of $M$ admits an orthogonal decomposition

$$TM = A \oplus B$$

where

$$g \text{ is flat on } B$$

then

$$(\Omega(M, TM), \nabla) \text{ is a } 2N\text{-dga for } N \geq \text{dim}(A).$$
Differential forms of depth $N$ on $\mathbb{R}^n$.

Fix an integer $N \geq 3$.

We construct the $(n(N - 1) + 1)$-differential graded algebra $\Omega_N(\mathbb{R}^n)$ of algebraic differential forms of depth $N$ on $\mathbb{R}^n$.

Let $x_1, \ldots, x_n$ be coordinates on $\mathbb{R}^n$.

For $0 \leq i \leq n$ and $0 \leq j < N$ let $d^j x_i$ be a variable of degree $j$.

We identify $d^0 x_i$ with $x_i$. 
The \((n(N - 1) + 1)\)-differential graded algebra \(\Omega_N(\mathbb{R}^n)\) is given by

\[
\Omega_N(\mathbb{R}^n) = \mathbb{R}[d^j x_i] / \langle d^j x_i d^k x_i \mid j, k \geq 1 \rangle
\]
as a graded algebras.

The \((n(N - 1) + 1)\)-differential \(d : \Omega_N(\mathbb{R}^n) \to \Omega_N(\mathbb{R}^n)\) is given by

\[
d(d^j x_i) = d^{j+1} x_i \quad \text{for} \quad 0 \leq j \leq N \quad \text{and} \quad d(d^{N-1} x_i) = 0.
\]

One can show that \(d\) is \((n(N - 1) + 1)\)-differential as follows:

It is easy to check that \(\Omega_N(\mathbb{R})\) is a \(N\)-dga.

If \(A\) is a \(N\)-dga and \(B\) is a \(P\)-dga, then \(A \otimes B\) is a \((N + P - 1)\)-dga.

\[
\Omega_N(\mathbb{R}^n) = \Omega_N(\mathbb{R}) \otimes^n.
\]
The construction above can be generalized to define the algebra

\[ \Omega_N(M) \]

of differential forms of depth \( N \) on each affine manifold.

Using a construction similar to Sullivan’s construction of algebraic differential forms on simplicial sets, one can construct for each simplicial set \( S \) a nilpotent differential graded algebra

\[ \Omega_N(S) \]

of differential forms of depth \( N \) on \( S \).
Difference forms of degree $N$ on affine discrete spaces.

There is a discrete analogue of the construction above. We define the depth $N$ analogue of Zeilberger’s difference forms on discrete affine space. Let $F(\mathbb{Z}^n, \mathbb{R})$ be the set of $\mathbb{R}$-valued maps on the lattice $\mathbb{Z}^n$.

Introduce variables $\delta^j m_i$ of degree $j$ for $1 \leq i \leq n$ and $1 \leq j < N$.

The graded algebra of difference forms of depth $N$ on $\mathbb{Z}^n$ is given by

$$D_N(\mathbb{Z}^n) = F(\mathbb{Z}^n, \mathbb{R}) \otimes \mathbb{R}[\delta^j m_i]/\langle \delta^j m_i \delta^k m_i \mid j, k \geq 1 \rangle.$$
A form $\omega \in D_N(\mathbb{Z}^n)$ can be written as

$$\omega = \sum_I \omega_I dm_I$$

where

$$I : \{1,..,n\} \rightarrow \mathbb{N}$$

is any map and

$$dm_I = \prod_{i=1}^{n} d^{I(i)} m_i.$$ 

The degree of $dm_I$ is

$$|I| = \sum_{i=1}^{n} I(i).$$
The finite difference $\Delta_i(g)$ of $g \in F(\mathbb{Z}^n, \mathbb{R})$ in the $i$-direction is given by

$$\Delta_i(g)(m) = g(m + e_i) - g(m),$$

where the vectors $e_i$ are the canonical generators of $\mathbb{Z}^n$ and $m \in \mathbb{Z}^n$.

The difference operator $\delta$ is defined for $1 \leq j \leq N - 2$ by the rules

$$\delta(g) = \sum_{i=1}^{n} \Delta_i(g)\delta m_i, \quad \delta(\delta^j m_i) = \delta^{j+1} m_i \quad \text{and} \quad \delta(\delta^{N-1} m_i) = 0.$$
It is not difficult to check that if \( \omega = \sum I \omega_I dm_I \), then

\[
\delta \omega = \sum_J (\delta \omega)_J dm_J
\]

where

\[
(\delta \omega)_J = \sum_{J(i)=1} (-1)^{|J<i|} \Delta_i \omega_{J-e_i} + \sum_{J(i)\geq 2} (-1)^{|J<i|} \omega_{J-e_i}.
\]

From the later formula we see that \((\delta \omega)_J\) is a linear combination of (differences of) functions \( \omega_K \) with \(|K| < |J|\). This fact implies that \( \delta \) is nilpotent, indeed, one can check that \( \delta^n(N-1)+1 = 0 \).
All together we have the following result.

\[ D_N(\mathbb{Z}^n) \] is a graded algebra and the difference operator \( \delta \) gives \( D_N(\mathbb{Z}^n) \) the structure of a \((n(N - 1) + 1)\)-complex.

One can check that \( \delta \) satisfies a twisted Leibnitz rule, so

\[ D_N(\mathbb{Z}^n) \]

is actually pretty close of being a \(N\)-dga.

Again using a Sullivan’s construction one associated an algebra of difference forms of depth \(N\) to each simplicial set. The algebra thus obtained is a (twisted) nilpotent-differential graded algebra.
Next we consider the deformation theory of nilpotent-differential graded algebras.

The \((N, M)\) Maurer-Cartan equation controls the deformations of a \(N\)-dga into a \(M\)-dga.

Let us review the construction of \((N, M)\) Maurer-Cartan equation.

Let \((A, d)\) be a \(N\)-differential graded algebra and \(e\) a degree one derivation of \(A\).

The \((N, M)\) Maurer-Cartan equation determines under which conditions \((A, d + e)\) is a \(M\)-differential graded algebra.
For \( s = (s_1, \ldots, s_n) \in \mathbb{N}^n \) we set \( l(s) = n \) and \( |s| = \sum_i s_i \).

For \( 1 \leq i < n \) we set \( s_{>i} = (s_{i+1}, \ldots, s_n) \).

For \( 1 < i \leq n \) we set \( s_{<i} = (s_1, \ldots, s_{i-1}) \) and \( s_{>n} = s_{<1} = \emptyset \).

\( \mathbb{N}^{(\infty)} \) denotes the set \( \bigcup_{n=0}^{\infty} \mathbb{N}^n \) where by convention \( \mathbb{N}^0 = \{\emptyset\} \).

For any \( c : A \to A \) set

\[ \hat{d}(c) = [d, c]. \]
For $s \in \mathbb{N}^{(\infty)}$ we set

$$e^{(s)} = e^{(s_1)} \ldots e^{(s_n)}$$

where

$$e^{(a)} = \hat{d}^a(e) \quad \text{if} \quad a \geq 1$$

$$e^{(0)} = e \quad \text{and} \quad e^\emptyset = 1.$$ 

For $M \in \mathbb{N}$ we let

$$E_M = \{ s \in \mathbb{N}^{(\infty)} : |s| + l(s) \leq M \}$$

and for $s \in E_M$ we define the integer $M(s)$ by

$$M(s) = M - |s| - l(s).$$
Recall that one can associated a discrete quantum mechanical system to each directed graph together with a weight attached to each of its edges.

Let us introduce a discrete quantum mechanical system given by the following data:
1. The set of vertices is $\mathbb{N}^{(\infty)}$.

2. There is a unique directed edge from vertex $s$ to vertex $t$ if and only if

$$t \in \{(0, s)\} \cup \{s\} \cup \{s + e_i \mid 1 \leq i \leq l(s)\}$$

where $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{N}^{l(s)}$. 

$$i-th$$
3. An edge $e$ with source $s(e)$ and target $t(e)$ is weighted according to the table

<table>
<thead>
<tr>
<th>$s(e)$</th>
<th>$t(e)$</th>
<th>$v(e_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>$(0, s)$</td>
<td>$1$</td>
</tr>
<tr>
<td>$s$</td>
<td>$s$</td>
<td>$(-1)^{</td>
</tr>
<tr>
<td>$s$</td>
<td>$s + e_i$</td>
<td>$(-1)^{</td>
</tr>
</tbody>
</table>

The set $P_M(\emptyset, s)$ consists of all paths

$$\gamma = (e_1, \ldots, e_M)$$

such that

$$s(e_1) = \emptyset \quad \text{and} \quad t(e_M) = s.$$
The weight $v(\gamma)$ of $\gamma \in P_M(\emptyset, s)$ is $v(\gamma) = \prod_{i=1}^{M} v(e_i)$.

The $(N, M)$ Maurer-Cartan equation is given by

$$\sum_{k=1}^{M-1} c_k d^k = 0$$

where

$$c_k = \sum_{\begin{subarray}{c} s \in E_M \\ M(s) = k \\ s_i < N \end{subarray}} c(s, M) e^{(s)} \quad \text{and} \quad c(s, M) = \sum_{\gamma \in P_M(\emptyset, s)} v(\gamma).$$
For example we have that

If $(A, d)$ is a 3-dga then $(A, d + e)$ is a 3-dga if and only if

$$(d^2(e) + d(e)e + e^3) + (d(e) + e^2)d + ed^2 = 0.$$  

If $(A, d)$ is a 3-dga then $(A, d + e)$ is a 4-dga if and only if

$$(e^4 + e^2d(e) + d(e)e^2 + d^2(e)e + ed^2(e) + (d(e))^2) + 2(e^2 + d(e))d^2 = 0.$$  

In deformation theory one also considers "infinitesimal" deformations

that is deformations of the form $d + te$

where $t$ is a formal variable such that $t \neq 0$ but $t^2 = 0$.  

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We set

\[ Par(k, N - k + 1) = \{ p = (p_1, \cdots, p_{N-k+1}) \mid \sum_{i=1}^{N-k+1} p_i = k \} \]

\[ w(p) = \sum_{i=1}^{N-k+1} (i - 1)p_i. \]

Let \((A, d)\) be a \(N\)-dga then \(e\) defines an infinitesimal deformation of the \(N\)-dga into the \(N\)-dga \((A[[t]]/(t^2), d + te)\) if and only if

\[ \sum_{k=0}^{N-1} \left( \sum_{p \in Par(k, N-k+1)} (-1)^{w(p)} \right) \hat{d}^{N-k-1}(e)d^{N-k-1} = 0. \]
N Lie algebroids.

We recall the definition of a Lie algebroid.

Let $E$ be a finite dimensional vector bundle. $E$ is a Lie algebroid if and only if $\Gamma(\wedge E^*)$ is a differential graded algebra. A differential on $\wedge E^*$ is the same as a degree one vector field $v$ on $E[-1]$ such that $v^2 = 0$.

Weakening the conditions on the vector field $v$ one gets various generalizations of the notion of Lie algebroids.

If one let $v$ of arbitrary degree, one get the notion of $L_\infty$ algebroid.

If instead of $v^2 = 0$ on impose higher order nilpotency we get the notion of $N$ Lie algebroids.
Higher nilpotency together with arbitrary degree yields the notion of $N L_\infty$ algebroids.

$N L_\infty$ algebroids have not been deeply studied so far.

An important subtlety is that the right higher nilpotency condition is not the naive $v \circ \ldots \circ v = 0$ as one could imagine.

Define the non-associative product on vector fields given by

$$v_i \partial_i \diamond w_j \partial_j = (v_i \partial_i w_j) \partial_j.$$

A $N$ Lie algebroid is a vector bundle $E$ together with a degree one derivation $d : \Gamma(\bigwedge E^*) \longrightarrow \Gamma(\bigwedge E^*)$, such that $d \diamond d \diamond \cdots \diamond d = 0$. 
It is not difficult to find examples of $N$ Lie algebroids. The simplest examples are the $N$ Lie algebras.

A $N$ Lie algebra is a vector space $\mathfrak{g}$ together with a degree one derivation $d$ on $\bigwedge \mathfrak{g}^*$ such that the $N$-th $\diamond$-composition of $d$ with itself vanishes.

We describe 3 Lie algebras in more familiar terms.

For integers $k_1, k_2, \ldots, k_l$ such that $k_1 + k_2 + \cdots + k_l = n$, we let $Sh(k_1, k_2, \cdots, k_l)$ be the set of permutations

$$\sigma : \{1, \cdots, n\} \rightarrow \{1, \cdots, n\}$$

such that $\sigma$ is increasing on the intervals $[k_i + 1, k_{i+1}]$ for $0 \leq i \leq l$, $k_0 = 1$ and $k_{l+1} = n$. 
Assume we are given a map \([\ , \ ] : \wedge^2 g \rightarrow g\).

The pair \((g, [\ , \ ])\) is a 3 Lie algebra if and only if for \(v_1, v_2, v_3, v_4 \in g\) we have

\[
\sum_{\sigma \in Sh(2,1,1)} \text{sgn}(\sigma) \left[ \left[ v_{\sigma(1)}, v_{\sigma(2)} \right], v_{\sigma(3)} \right] v_{\sigma(4)} =
\]

\[
= \sum_{\sigma \in Sh(2,2)} \text{sgn}(\sigma) \left[ \left[ v_{\sigma(1)}, v_{\sigma(2)} \right], \left[ v_{\sigma(3)}, v_{\sigma(4)} \right] \right]
\]
We closed this talk mentioning that there is a higher depth analogue of the notion of $N$ infinity algebras which we called

$$A_N^\infty \, - \, \text{algebras.}$$

An $A_N^\infty$-algebra is a graded vector space $A$ together with a sequence of degree one maps

$$m_k : A[1]^\otimes k \rightarrow A[1]$$

such that the associated coderivation $\delta = \sum m_i$ on $T(A[1])$ satisfies

$$\delta \diamond \delta \ldots \diamond \delta = 0.$$
The condition $\delta \diamond \delta = 0$ defining $A^2_\infty$-algebras is the familiar condition for $A_\infty$-algebras:

$$\sum_{r+s+t=n} m_{r+1+t} \circ (1 \otimes r \otimes m_s \otimes 1 \otimes t) = 0.$$ 

The condition $\delta \diamond \delta \diamond \delta = 0$ defining an $A^3_\infty$-algebra is

$$\sum_{a+b+c+d+e=n} m_{a+e+1} \circ (1 \otimes a \otimes m_{b+d+1} \otimes 1 \otimes e) \circ (1 \otimes a+b \otimes m_c \otimes 1 \otimes d+e) +$$

$$m_{a+b+c+e+1} \circ (1 \otimes a \otimes m_b \otimes 1 \otimes c+e+1) \circ (1 \otimes a+b+c \otimes m_d \otimes 1 \otimes e) +$$

$$m_{a+c+d+e+1} \circ (1 \otimes a+c+1 \otimes m_d \otimes 1 \otimes e) \circ (1 \otimes a \otimes m_b \otimes 1 \otimes c+d+e) = 0$$
It becomes rather cumbersome to write an explicit formula for the condition $\delta \diamond \delta \ldots \diamond \delta = 0$ for $N \geq 4$.

We rewrite that condition in terms of trees.

Let $RT^n_l$ be the set of isomorphism classes of rooted planar trees with $l$ leaves and $n$ internal vertices.

For example the following trees are in $RT^3_{16}$:
The condition $\delta \diamond \delta \ldots \diamond \delta = 0$ holds if and only if for each $l \in \mathbb{N}_+$

$$\sum_{\Gamma \in RT_i^N} m_\Gamma = 0,$$

where $m_\Gamma$ is defined by a procedure similar to that use in the graphical definition of $A_\infty$-algebras putting the $m_s$ operator on each vertex with $s$ incoming edges attached to it.
¡THANKS!