Symmetric Brace Algebras

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This is joint work with Martin Markl.

Recall the definition of a (non-symmetric) brace algebra as given by Gerstenhaber and Voronov:

Definition 1. *A brace algebra is a graded vector space U together with a collection of degree 0 multilinear braces $x, x_1, \ldots, x_n \longmapsto x\{x_1, \ldots, x_n\}$ that satisfy the identities*

$$x\{\ \} = x$$

and

$$x\{x_1, \ldots, x_m\}\{y_1, \ldots, y_n\} =$$
$$\sum \epsilon \cdot x\{y_1, \ldots, y_{i_1}, x_1\{y_{i_1+1}, \ldots, y_{j_1}\}, y_{j_1+1}, \ldots, y_{i_m}, x_m\{y_{i_m+1}, \ldots, y_{j_m}\}, y_{j_m+1}, \ldots, y_n$$

where the sum is taken over all sequences $0 \leq i_1 \leq j_1 \leq \cdots \leq i_m \leq j_m \leq n$ and where ϵ is the Koszul sign of the permutation

$$(x_1, \ldots, x_m, y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_{i_1}, x_1, y_{i_1+1}, \ldots, y_{j_1}, y_{j_1+1}, \ldots, y_{i_m}, x_m, y_{i_m+1}, \ldots, y_{j_m}, y_{j_m+1}, \ldots, y_n)$$

of elements of U.

We consider a symmetric version of these braces:

Definition 2. *A symmetric brace algebra is a graded vector space B together with a collection of degree 0 multilinear braces $x\langle x_1, \ldots, x_n\rangle$ that are graded symmetric in $x_1, \ldots, x_n$ and satisfy the identities*

$$x\langle\ \rangle = x$$

and

$$x\langle x_1, \ldots, x_m\rangle\langle y_1, \ldots, y_n\rangle =$$
$$\sum \epsilon \cdot x\langle x_1\langle y_{i^1_1}, \ldots, y_{i^1_{t_1}}\rangle, x_2\langle y_{i^2_1}, \ldots, y_{i^2_{t_2}}\rangle, \ldots, x_m\langle y_{i^m_1}, \ldots, y_{i^m_{t_m}}\rangle, y_{i^{m+1}_1}, \ldots, y_{i^{m+1}_{t_{m+1}}}\rangle$$

where the sum is taken over all unshuffle sequences

$$i^1_1 < \cdots < i^1_{t_1}, \ldots, i^{m+1}_1 < \cdots < i^{m+1}_{t_{m+1}}$$

of $\{1, \ldots, n\}$ and where ϵ is the Koszul sign of the permutation

$$(x_1, \ldots, x_m, y_1, \ldots, y_n) \longmapsto (x_1, y_{i^1_1}, \ldots, y_{i^1_{t_1}}, x_2, y_{i^2_1}, \ldots, y_{i^2_{t_2}}, \ldots, x_m, y_{i^m_1}, \ldots, y_{i^m_{t_m}}, y_{i^{m+1}_1}, \ldots, y_{i^{m+1}_{t_{m+1}}})$$

of elements of B.

Some examples with signs suppressed:
Non-symmetric brace relation

$$x\{x_1, x_2\}\{y\} = x\{y, x_1, x_2\} + x\{x_1, y, x_2\} + x\{x_1, x_2, y\}$$
$$+x\{x_1\{y\}, x_2\} + x\{x_1, x_2\{y\}\}$$

and the corresponding symmetric brace relation

$$x\langle x_1, x_2\rangle\langle y\rangle = x\langle x_1, x_2, y\rangle + x\langle x_1\langle y\rangle, x_2\rangle + x\langle x_1, x_2\langle y\rangle\rangle$$

Another example

$$x\{x_1\}\{y_1, y_2\} = x\{x_1, y_1, y_2\} + x\{y_1, x_1, y_2\} + x\{y_1, y_2, x_1\}$$
$$+x\{x_1\{y_1\}, y_2\} + x\{y_1, x_1\{y_2\}\} + x\{x_1\{y_1, y_2\}\}$$

and the symmetric version

$$x\langle x_1\rangle\langle y_1, y_2\rangle = x\langle x_1, y_1, y_2\rangle + x\langle x_1\langle y_1\rangle, y_2\rangle + x\langle x_1\langle y_2\rangle, y_1\rangle + x\langle x_1\langle y_1, y_2\rangle\rangle$$

The motivating example for brace algebra structures is the following: let V be a graded vector space and consider the graded vector space $B_*(V)$ where

$$B_s(V) := \bigoplus_{p-k+1=s} Hom(V^{\otimes k}, V)_p$$

where $Hom(V^{\otimes k}, V)_p$ denotes the space of k-multilinear maps of degree p . Given $f \in Hom(V^{\otimes N}, V)$ and $g_i \in Hom(V^{\otimes a_i}, V)$, where q_i is the degree of g_i as a map, define

$$f\{g_1, \dots, g_n\} \in Hom(V^{\otimes r}, V)_{p+q_1+\cdots+q_n}$$

where $r = a_1 + \dots a_n + N - n$ by

$$f\{g_1, \dots, g_n\} = \sum_{k_0+\dots k_n = N-n} (-1)^{\beta} f(1^{\otimes k_0} \otimes g_1 \otimes 1^{\otimes k_1} \otimes \cdots \otimes 1^{\otimes k_{n-1}} \otimes g_n \otimes 1^{\otimes k_n}),$$

where

$$\beta = \sum_{j<i} [a_i - 1]\,[k_j + a_j] + \sum_i (N-i)\, q_i + \sum_{j<i} q_i\, a_j.$$

In this example, suppose that we have a collection of maps

$$\mu_k \in Hom(V^{\otimes k}, V)_{k-2} \in B_{-1}(V).$$

If we let $\mu = \mu_1 + \mu_2 + \dots$, then an A_∞ algebra structure on V may be described by the brace relation $\mu\{\mu\} = 0$. For example (ignoring signs)

$$\mu\{\mu\}(x, y, z) = \mu_1\{\mu_3\}(x, y, z) + \mu_2\{\mu_2\}(x, y, z) + \mu_3\{\mu_1\}(x, y, z) =$$

$$\mu_1(\mu_3(x, y, z)) + \mu_2(\mu_2(x, y), z) + \mu_2(x, \mu_2(y, z))$$

$$+\mu_3(\mu_1(x), y, z) + \mu_3(x, \mu(y), z) + \mu_3(x, y, \mu(z)) = 0,$$

i.e. μ_2 is a homotopy associative operation.

We now consider examples of symmetric brace algebras by "symmetrizing" the previous examples. Let $Hom(V^{\otimes k}, V)^{as}_p$ be the space of k-multilinear maps of degree p that are antisymmetric (or alternating). Consider the graded vector space $B_*(V)$ where

$$B_s(V) := \bigoplus_{p-k+1=s} Hom(V^{\otimes k}, V)^{as}_p.$$

Given $f \in Hom(V^{\otimes k}, V)^{as}_p$ and $g_i \in Hom(V^{\otimes a_i}, V)^{as}_{q_i}$, $1 \leq i \leq n$, define the symmetric brace $f\langle g_1, \ldots, g_n\rangle \in Hom(V^{\otimes r}, V)^{as}_{p+q_1+\cdots+q_n}$, where $r := a_1 + \cdots + a_n + k - n$, by

$$f\langle g_1, \ldots, g_n\rangle(v_1, \ldots, v_r) := \sum (-1)^{\delta}\chi \cdot f(g_1 \otimes \cdots \otimes g_n \otimes 1\!\!1^{\otimes k-n})(v_{i_1}, \ldots, v_{i_r})$$

with the summation taken over all unshuffles

$$i_1 < \cdots < i_{a_1}, i_{a_1+1} < \ldots < i_{a_1+a_2}, \ldots, i_{a_1+\cdots+a_k+1} < \cdots < i_r,$$

of elements of V, where χ is the antisymmetric Koszul sign of the permutation

$$(v_1, \ldots, v_r) \longmapsto (v_{i_1}, \ldots, v_{i_r})$$

and

$$\begin{aligned}\delta &= (k-1)q_1 + (k-2+a_1)q_2 + \cdots + (k-n+a_1+\cdots+a_{n-1})q_n \\ &\quad + \textstyle\sum_{1\leq i<j\leq n} a_i a_j + (n-1)a_1 + (n-2)a_2 + \cdots + a_{n-1}.\end{aligned}$$

In this example, suppose that we have maps

$$l_k \in Hom(V^{\otimes k}, V)^{as}_{k-2} \in B_{-1}(V).$$

If we let $l = l_1 + l_2 + \ldots$, then an L_∞ algebra structure on V is given by the symmetric brace relation $l\langle l\rangle = 0$.

Example (again, ignoring signs): $l\langle l\rangle(x, y, z) =$

$$l_1\langle l_3\rangle(x, y, z) + l_2\langle l_2\rangle(x, y, z) + l_3\langle l_1\rangle(x, y, z) =$$

$$l_1(l_3(x, y, z)) + l_2(l_2(x, y), z) + l_2(l_2(x, z), y) + l_2(l_2(y, z), x)$$

$$l_3(l_1(x), y, z) + l_3(l_1(y), x, z) + l_3(l_1(z), x, y) = 0.$$

Symmetrization Theorems

1. Given a brace algebra structure on the graded vector space U, then

$$f\langle g_1, \ldots, g_n\rangle = \sum_{\sigma\in S_n} \epsilon \cdot f\{g_{\sigma(1)}, \ldots, g_{\sigma(n)}\}$$

is a symmetric brace algebra structure on U where ϵ is the Koszul sign of the permutation σ.

2. Given $f : V^{\otimes n} \to V$, and let $as(f)$ denote the map

$$as(f)(v_1, \ldots, v_n) = \sum_{\sigma\in S_n} \chi f(v_{\sigma(1)}, \ldots, v_{\sigma(n)}).$$

Let $f, g_1, \ldots, g_n \in Hom(V^{\otimes *}, V)$. Then

$$\sum_{\sigma\in S_n} \epsilon \cdot as(f\{g_{\sigma(1)}, \ldots, g_{\sigma(n)}\}) = as(f)\langle as(g_1), \ldots, as(g_n)\rangle.$$

3. Corollary: Consider the symmetric brace algebra structure on $\bigoplus Hom(V^{\otimes *}, V)$ given by the symmetrization of the non-symmetric brace structure given earlier. Let $\bigoplus Hom(V^{\otimes *}, V)^{as}$ have the symmetric brace algebra structure given above. Then

$$as : \bigoplus Hom(V^{\otimes *}, V) \to \bigoplus Hom(V^{\otimes *}, V)^{as}$$

is a homomorphism of symmetric brace algebras.

4. Also as a Corollary, we have
The anti-symmetrization $l := as(\mu)$ of an A_∞ algebra structure μ yields an L_∞ algebra structure.

Proof: Given $\mu\{\mu\} = 0$. We have

$$0 = as(\mu\{\mu\}) = as(\mu)\langle as(\mu)\rangle = l\langle l\rangle.$$

MOTIVATION FROM A PHYSICS PROBLEM

So let L be the graded vector space with $L_0 = \Xi$, $L_{-1} = \Phi$, $L_n = 0$, $n \neq 0, -1$.

Consider the symmetric brace algebra structure on $B_*(L) = Hom(L^{\otimes *}, L)^{as}$.

Construct a bracket on $Hom(L^{\otimes *}, L)^{as}$ so that the existance of an $L\infty$ structure on L is equivalent to this's bracket satisfying the Jacobi identity on $Hom(\Phi^{\otimes *}, \Xi)^{as}$:

Fix two maps, ∇ and Υ in $Hom(L^{\otimes *}, L)^{as}$ we require that

1) ∇ has values in Φ and is the zero map when the number of inputs from Ξ is not equal to 1.

2) Υ takes values in Ξ and is the zero map when the number of inputs from Ξ is not equal to 2.

e.g. $Hom(\Xi, Hom(\Lambda^*\Phi, \Phi)) \ni \delta \longleftrightarrow \nabla \in Hom(\Xi \otimes \Lambda^*\Phi, \Phi)$

$Hom(\Xi \wedge \Xi, Hom(\Lambda^*\Phi, \Xi)) \ni C \longleftrightarrow \Upsilon \in Hom(\Xi \wedge \Xi \otimes \Lambda^*\Phi, \Xi)$

Note that $\nabla, \Upsilon \in B_{-1}(L)$. For $\alpha, \beta \in B_*(L)$ define a degree -1 bracket

$$[\alpha, \beta] = \alpha\langle\nabla\langle\beta\rangle\rangle + (-1)^{\alpha\beta}\beta\langle\nabla\langle\alpha\rangle\rangle + \Upsilon\langle\alpha, \beta\rangle.$$

Theorem (L·M)

The above bracket, restricted to $Hom(\Phi^{\otimes *}, \Xi)^{as} \subset B_{-1}(V)$ satisfies the Jacobi identity if and only if

$$\nabla\langle\nabla\rangle + \nabla\langle\Upsilon\rangle = 0$$

and $\Upsilon\langle\nabla\rangle + \Upsilon\langle\Upsilon\rangle = 0$

Cor: if this bracket satisfies the Jacobi identity, then the map $l = \nabla + \Upsilon$ gives an L_∞ structure for L

Proof:

$$l\langle l\rangle = (\nabla+\Upsilon)\langle\nabla+\Upsilon\rangle$$

$$= (\nabla\langle\nabla\rangle + \nabla\langle\Upsilon\rangle + (\Upsilon\langle\nabla\rangle + \Upsilon\langle\Upsilon\rangle$$

$$= 0$$