Orbifold Cohomology for Global Torus Quotients

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WARNING: These are lecture notes!

<u>Introductory Example.</u>

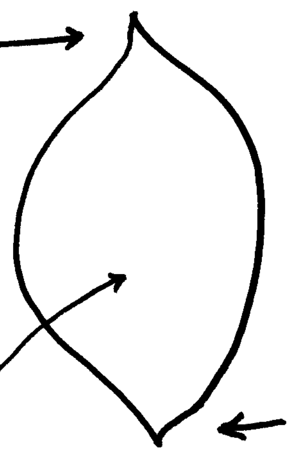
Consider the topological quotient of S^3 by S^1 where the action is:

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$$

S^1 acts on S^3 by:

$$\lambda \cdot (z_1, z_2) = (\lambda^2 z_1, \lambda^3 z_2).$$

The quotient space S^3/S^1 is topologically a 2-sphere, but:



The orbit of points $(z_1, 0)$ form the North pole, with isotropy $\mathbb{Z}_2$: $1, -1$ both fix $(z_1, 0)$

it really looks like a lemon.

The orbit of a generic point (z_1, z_2) $(z_1, z_2 \neq 0)$ is smooth in the quotient.

The orbit of pts $(0, z_2)$ form the South pole, with isotropy $\mathbb{Z}_3$: $1, e^{2\pi i/3}, e^{4\pi i/3}$ all fix $(0, z_2)$.

The lemon is not the global quotient of any space by a finite group.

The inertial cohomology ring.

How can we devise an invariant to tell the difference between the smooth S^2 and the lemon?

Introduce the inertial cohomology ring.

Let M be a manifold with a locally free T action (here, T is a torus, and loc. free means finite isotropy).

Then as a (graded) vector space:

$$NH_T^{*,\diamond}(M) := \bigoplus_{g \in T} H^*(M^g/T)$$

$$M^g = \{m \in M \mid g \cdot m = m\}$$

(Many $M^g = \phi$)

- The grading $*$ is shifted from left hand side to right hand side. We won't discuss this
- The grading $\diamond$ is over the group elements. We won't discuss this either (both gradings are related to the product structure, which we won't have time for).

In the case of $M = S^3$ and $T = S^1$ with the action as before,

$$NH^{*,\diamond}_{S^1}(S^3) = \underbrace{H^*(S^3/S^1)}_{g=1} \oplus \underbrace{H^*(\{(z_1,0)\}/S^1)}_{g=-1}$$

$$\oplus \underbrace{H^*(\{(0,z_2)\}/S^1)}_{g=e^{2\pi i/3}} \oplus \underbrace{H^*(\{(0,z_2)\}/S^1)}_{g=e^{4\pi i/3}}$$

Equivariant Cohomology.

$$H^*_T(M) := H^*(M \times_T ET)$$

where ET is a contractible space on which T acts freely, and $M \times_T ET := (M \times ET)/T$

Note: $\underset{\text{(rationally)}}{H^*_T(M) =} H^*(M/T)$ when T acts freely.

We may rewrite:

$$NH^{*,\diamond}_T(M) := \bigoplus_{g \in T} H^*_T(M^g)$$

(when T acts on M locally freely).

Chen & Ruan describe a funny ring structure on this space, but it's not in practice easy to compute.

Idea: Move away from locally free actions

Let Y be a Hamiltonian T-space. (T = compact abelian Lie group)
This means Y is symplectic (~~with~~ has a closed, non-degenerate 2-form ω on it).

and if $\xi \in \mathrm{Lie}(T)$ is a vector in the tangent space, ~~ξ~~ generates ~~the~~ the vector field X_ξ on Y, then $\exists$ function ϕ^ξ on Y such that

$$\omega(X_\xi, \cdot) = d\phi^\xi .$$

In particular $\phi : Y \longrightarrow \mathrm{Lie}(T)^*$

defined by

$$\langle \phi(p), \xi \rangle \text{ is } \phi^\xi(p)$$

is ~~the~~ a moment map for $T \curvearrowright Y$.

Hamiltonian T-spaces do **not** have locally free actions: if Y is compact the functions ϕ^ξ have max/min on Y

$\Rightarrow d\phi^\xi = 0$ at these pts

$\Rightarrow$ by nondegen. of ω, $X_\xi = 0$ at these points

$\Rightarrow$ the $S^1 \subset T$ generated by (rational) ξ fixes the crit. point of ϕ^ξ.

NOT LOCALLY FREE

However, the inertial cohomology ring is still well-defined using equivt cohomology

$$NH_T^{*,\diamond}(Y) := \bigoplus_{g\in T} H_T^*(Y^g).$$

In the case that $T \curvearrowright Y$ locally freely, this is the ~~Ea~~ Chen-Ruan cohomology of Y/T.

In the case of $T \curvearrowright Y$ in a Hamiltonian fashion, this is something else.

If $T \curvearrowright Y$ in a Hamiltonian fashion with fixed points Y^T, then

$$H_T^*(Y) \hookrightarrow H_T^*(Y^T)$$ is an INJECTION.

If Y^T is isolated,

$$H_T^*(Y) \hookrightarrow \bigoplus_{p\in Y^T} H_T^*(p)$$

$$= \bigoplus_{p\in Y^T} \mathbb{Q}[u_1,\dots,u_d]$$ where $d = \dim T$

(since $H_T^*(p) = H^*(p\times_T ET) = H^*(BT) \cong H^*(\underbrace{\mathbb{CP}^\infty\times\cdots\times\mathbb{CP}^\infty}_{d\text{ copies}})$)

Contrast this to ordinary cohomology where there is only degree 0 cohomology on Y^T, when isolated.
AND NO POSSIBLE INJECTION IN GENERAL

Let
$$\phi : Y \longrightarrow \mathrm{Lie}(T)^*$$
be a moment map. Assume 0 is a regular value.

Then $\phi^{-1}(0) \subset Y$ is a submanifold with a LOCALLY FREE T-action.

<u>Ex</u>

$Y = S^2$, $\omega =$ volume form

$S^1 \curvearrowright Y$ by spinning it on z-axis, fixing N and S poles

$$\phi : Y \to \mathbb{R}$$ moment map is height function.

$$S^2 \xrightarrow[\text{function}]{\text{ht}}$$


0 is a regular value.

$\phi^{-1}(0)$ is equator. It has a free S^1 action.

The symplectic <u>reduction</u>
$$Y/\!/_T := \phi^{-1}(0)/_T$$
is an orbifold.

Theorem (G-, Holm, Knutson): There is a natural surjection

$$NH_T^{*,\diamond}(Y) \longrightarrow NH_T^{*,\diamond}(\phi^{-1}(0))$$

SURJECTION

$$\cong H_{CR}^{*}(Y/\!/T)$$

as rings from the inertial cohomology of a Hamiltonian T-space Y to the Chen-Ruan cohomology of the symplectic quotient $Y/\!/T$.

If $Y = T^*\mathbb{C}^n$ is Hyperkähler, and $Y/\!/\!/\!/T$ is a hypertoric variety, then

Thm (G-, Harada)

$$NH_T^{*,\diamond}(Y) \longrightarrow H_{CR}^{*}(Y/\!/\!/\!/T)$$

is also a surjection.

By the way, the Lemon (our original example) is a symplectic reduction of $\mathbb{C}^2$ by an S^1 action:

$$\lambda \cdot (z_1, z_2) = (\lambda^2 z_1, \lambda^3 z_2).$$

The moment map $\phi : \mathbb{C}^2 \to \mathbb{R}$ is given by

$$(z_1, z_2) \longmapsto |z_1|^2 + |z_2|^2 - 1.$$

Then lemon $= \phi^{-1}(0)/S^1$.