SYMMETRIZATION OF BRACE ALGEBRAS

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Abstract. We show that the symmetrization of a brace algebra structure yields the structure of a symmetric brace algebra. We also show that the symmetrization of the natural brace structure on \( \bigoplus_{k \geq 1} \text{Hom}(V^\otimes k, V) \) coincides with the natural symmetric brace structure on \( \bigoplus_{k \geq 1} \text{Hom}(V^\otimes k, V)^{as} \), the space of antisymmetric maps \( V^\otimes K \rightarrow V \).

1. Introduction

Brace algebras were first studied in the context of multilinear operations on the Hochschild complex of an associative algebra [3, 2, 1]. Symmetric brace algebras, in which the brace operations possess the property of graded symmetry, were subsequently introduced in [5]. Just as one may construct \( L_\infty \) algebra structures by antisymmetric \( A_\infty \) algebra structures [4], we show in this note that the symmetrization of a brace algebra structure yields a symmetric brace algebra structure. We prove in Section 5 that

\[
\langle f(g_1, \ldots, g_n) \rangle := \sum_{\sigma \in S_n} \epsilon(\sigma) f\{g_{\sigma(1)}, \ldots, g_{\sigma(n)}\}
\]

where \( \langle, \rangle \) and \( \{,\} \) denote symmetric and non-symmetric braces respectively.

The motivating example of a brace algebra is \( \bigoplus_{k \geq 1} \text{Hom}(V^\otimes k, V) \), and the fundamental example of a symmetric brace algebra is the subspace of anti-symmetric maps, \( \bigoplus_{k \geq 1} \text{Hom}(V^\otimes k, V)^{as} \).

In Section 6, we show that these algebras are related by

\[
\sum_{\sigma \in S_n} \epsilon(\sigma) as(f\{g_{\sigma(1)}, \ldots, g_{\sigma(n)}\}) = as(f)\langle g_1, \ldots, g_n \rangle,
\]

where \( as(f)(v_1, \ldots, v_k) := \sum_{\sigma \in S_k} (-1)^{\epsilon(\sigma)} f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \) and \( \epsilon(\sigma) \) is just the Koszul sign of the permutation.

In Sections 2 and 3, we review the definitions and fundamental examples of brace algebras and symmetric brace algebras respectively. Section 4 contains a collection of technical lemmas that are needed to prove the main theorems in the final two sections.

2. Brace Algebras

Definition 1. A brace structure on a graded vector space consists of a collection of degree 0 multilinear braces \( x, x_1, \ldots, x_n \mapsto x\{x_1, \ldots, x_n\} \) which satisfy the identity, \( x\{\} = x \), and in which \( x\{x_1, \ldots, x_n\}\{y_1, \ldots, y_r\} \) is equal to

\[
\sum \epsilon \cdot x\{y_1, \ldots, y_{i_1}, x_1\{y_{i_1+1}, \ldots, y_{j_1}\}, y_{j_1+1}, \ldots, y_{i_2}, x_2\{y_{i_2+1}, \ldots, y_{j_2}\}, y_{j_2+1}, \ldots, y_r\}.
\]

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In the above formula, the sum is over all sequences $0 \leq i_1 \leq j_1 \leq \cdots \leq i_n \leq j_n \leq r$, and $\epsilon$ is the Koszul sign of the permutation which maps $(x_1, \ldots, x_n, y_1, \ldots, y_r)$ to 

$$(y_1, \ldots, y_{i_1}, x_1, y_{i_1+1}, \ldots, y_{i_2}, y_{j_1+1}, \ldots, y_{i_n}, x_n, y_{i_n+1}, \ldots, y_{j_n+1}, \ldots, y_r).$$

The motivating example for a brace algebra structure is the space $\text{Hom}(V^\otimes N, V)$ with the natural brace operation of degree $-n$ given by the composition 

$$f\{g_1, \ldots, g_n\} = \sum_{k_0 + \cdots + k_n = N-n} f(1^{\otimes k_0} \otimes g_1 \otimes 1^{\otimes k_1} \otimes \cdots \otimes g_n \otimes 1^{\otimes k_n}),$$

where $f \in \text{Hom}(V^\otimes N, V)$. This operation arises from the endomorphism operad of $V$ considered in [1]. This operation was also utilized in the context of the Hochschild complex of the associative algebra $V$ in [3] and [2]. After a regrading, this example may be regarded as a special case of the following

**Example 2.** Let $V$ be a graded vector space and consider the graded vector space $B_*(V)$ where 

$$B_*(V) := \bigoplus_{p-k+1=s} \text{Hom}(V^\otimes k, V)_p$$

and where $\text{Hom}(V^\otimes k, V)_p$ denotes the space of $k$-multilinear maps of degree $p$. Given $f \in \text{Hom}(V^\otimes N, V)_p$ and $g_i \in \text{Hom}(V^\otimes a_i, V)_{q_i}$, define $f\{g_1, \ldots, g_n\} \in \text{Hom}(V^\otimes r, V)_{p+q_1+\cdots+q_n}$, where $r = a_1 + \cdots + a_n + N-n$ by 

$$f\{g_1, \ldots, g_n\} = \sum_{k_0 + \cdots + k_n = N-n} (-1)^{\beta} f(1^{\otimes k_0} \otimes g_1 \otimes 1^{\otimes k_1} \otimes \cdots \otimes 1^{\otimes k_{n-1}} \otimes g_n \otimes 1^{\otimes k_n}),$$

where 

$$\beta = \sum_{j<i} [a_i - 1][k_j + a_j] + \sum_i (N-i)q_i + \sum q_i a_j.$$

**Remark 3.** In Example 2, suppose that there exists a collection of maps $\mu_k \in \text{Hom}(V^\otimes k, V)_{k-2} \in B_{-1}(V)$.

If we let $\mu = \mu_1 + \mu_2 + \ldots$, then an $A_\infty$ algebra structure on $V$ may be described by the brace relation $\mu(\mu) = 0$ [5].

3. **Symmetric Brace Algebras**

**Definition 4.** An $n$-unshuffle of $N$ elements is a partition $\sum_{i=1}^n a_i = N$ and a permutation $\gamma \in S_N$ such that 

$$\gamma(1) < \cdots < \gamma(a_1), \gamma(1+a_1) < \cdots < \gamma(a_2+a_1), \ldots, \gamma(1 + \sum_{i=1}^{n-1} a_i) < \cdots < \gamma(N).$$

**Definition 5.** A symmetric brace algebra is a graded vector space together with a collection of degree zero multilinear braces $f\{g_1, \ldots, g_n\}$ which are graded symmetric in $g_1, \ldots, g_n$. In a symmetric brace algebra, it is also required that $f(\overline{)} = f$, and that $f\{g_1, \ldots, g_n\}\langle x_1, \ldots, x_r \rangle$ be equal to 

$$\sum_{\gamma \text{ is an unshuffle of } (n+1)} \epsilon \cdot f\{g_1\langle x_{\gamma(1)}, \ldots, x_{\gamma(a_1)} \rangle, \ldots, g_n\langle x_{\gamma(1 + \sum_{i=1}^{n-1} a_i)}, \ldots, x_{\gamma(\sum_{i=1}^a a_i)} \rangle, x_{\gamma(1 + \sum_{i=1}^{n-1} a_i)}, \ldots, x_{\gamma(r)} \rangle.$$
where $\epsilon$ is the Koszul sign of the permutation which maps $(g_1,\ldots,g_n,x_1,\ldots,x_r)$ to 
$$\left( g_1,x_{\gamma(1)},\ldots,x_{\gamma(a_1)};g_2,\ldots,x_{\gamma(1+\sum_{i=1}^{n-1}a_i)};\ldots,x_{\gamma(\sum_{i=1}^{n}a_i)};g_n,\ldots,x_{\gamma(r)} \right).$$

Just as with brace algebras, the fundamental example of a symmetric brace algebra is provided by the space of antisymmetric maps of degree $p$, $\text{Hom}(V^\otimes k,V)^{as}_p$. To be precise, we have

**Example 6.** Let $V$ be a graded vector space and $B_s(V)$ be the graded vector space given by
$$B_s(V) = \bigoplus_{p-k+1=s} \text{Hom}(V^\otimes k,V)^{as}_p.$$ 

Given $f \in \text{Hom}(V^\otimes k,V)^{as}_p$ and $g_i \in \text{Hom}(V^\otimes a_i,V)^{as}_{q_i}$, $1 \leq i \leq n$, define the symmetric brace
$$f\langle g_1,\ldots,g_n \rangle(x_1,\ldots,x_r) = (-1)^{\delta} \sum_{\gamma \text{ is an \ unshuffle}} \chi(\gamma) f(g_1 \otimes \cdots \otimes g_n \otimes 1^{\otimes N-n})(x_{\gamma(1)},\ldots,x_{\gamma(r)}),$$

where
$$\delta = \sum_i (N-i)q_i + \sum_{j<i} q_ia_j + \sum_{j<i} a_ia_j + \sum_i (n-i)a_i,$$

and $\chi$ is the antisymmetric Koszul sign of the permutation $\gamma$.

**Remark 7.** Suppose that in Example 6 we have maps
$$l_k \in \text{Hom}(V^\otimes k,V)^{as}_{p-2} \in B_{-1}(V).$$

If we let $l = l_1 + l_2 + \ldots$, then an $L_\infty$ algebra structure on $V$ is given by the symmetric brace relation $l(l) = 0$.

4. SOME LEMMAS

Although the expressions in this paper involve many sums, permutations, and antisymmetrizations, we will be able to simplify things considerably with the help of the following lemmas. Lemma (8) provides a decomposition of $as(f)$ which will be useful later.

**Lemma 8.** $as(f) = f \circ \Phi_{nm} \circ \Psi_{n} \circ \Theta_{m}$ \  $\forall f \in \text{Hom}(V^\otimes n+m,V)$, where
- $\Theta_{m}(y_1,\ldots,y_n,z_1,\ldots,z_m) = \sum_{\pi \in S_n} \chi(\pi)(y_1,\ldots,y_n,z_{\pi(1)},\ldots,z_{\pi(m)}),$ 
- $\Psi_{n}(y_1,\ldots,y_n,z_1,\ldots,z_m) = \sum_{\sigma \in S_n} \chi(\sigma)(y_{\sigma(1)},\ldots,y_{\sigma(n)},z_1,\ldots,z_m),$ 
- $\Phi_{nm}(y_1,\ldots,y_n,z_1,\ldots,z_m) = \sum_{k_0+\cdots+k_n=m}(-1)^{\eta} (z_{k_0},y_1,z_{1+k_0},\ldots,y_n,z_{1+k_0+\cdots+k_{n-1}},\ldots,z_m),$ 
- $\eta = \sum_{i=1}^{n} \{ y_i \left[ z_1 + \cdots + z_{(k_0+k_1+\cdots+k_{i-1})} \right] + (n-i)k_i \}.$
Proof. Since \( \Psi_n \) does all permutations of the first \( n \) inputs, \( \Theta_m \) provides all permutations of the last \( m \) inputs, and \( \Phi_{nm} \) distributes the last \( n \) variables between the first \( m \) in every possible way, the composition is clearly a sum of all permutations of the original \( n + m \) variables. A moment’s reflection also reveals that the sign of each summand in the composition is the Koszul sign together with the sign of the permutation. \( \square \)

Lemma 9. If \( N = a_1 + \cdots + a_n \), then \( \sum_{\pi \in S_N} \chi(\pi)(x_{\pi(1)}, \ldots, x_{\pi(N)}) \) is equal to

\[
\sum_{\gamma \in \chi_{\pi_1 \ldots \pi_n}^{S_{a_1} \ldots S_{a_n}} \text{unshuffle}} \chi(\gamma) \left( \sum_{\pi_j \in S_{a_j}} \chi(\pi_j) \prod_{i=1}^{n} x_{\gamma(\pi_j(1))} \cdots x_{\gamma(\pi_j(a_j))} \prod_{i=1}^{n} x_{\gamma(\pi_j(a_j)+1)} \cdots x_{\gamma(\pi_j(a_j)+n)} \right).
\]

Proof. Clearly, the right hand side is the sum of distinct permutations of the \( x \) terms with the correct sign. Furthermore, since there are \( \frac{N!}{a_1! \cdots a_n!} \) unshuffles \( \gamma \) and \( (a_i)! \) permutations \( \pi_j \), there are \( N! \) summands in the right hand side, which agrees with the number of summands on the left hand side. \( \square \)

Lemma 10. Suppose \( k_0 + a_1 + k_1 + \cdots + a_n + k_n = r \), \( \sigma \in S_n \), and \( \pi \in S_r \). Let \( A = a_1 + \cdots + a_n \), denote \( X_i = x_{\pi(1+a_1+\cdots+a_{i-1})}, \ldots, x_{\pi(a_1+\cdots+a_i)} \), and also denote \( X_{\pi} = x_{\pi(1+A)}, \ldots, x_{\pi(k_0+A)}, x_{\pi(1)}, \ldots, x_{\pi(1+k_0+A)}, \ldots, x_{\pi(n)}, \ldots, x_{\pi(1+k_0+\cdots+k_{n-1}+A)}, \ldots, x_{\pi(r)}. \)

Then we can define \( \hat{\pi} \in S_r \) by

\[
\hat{\pi}(i) = \begin{cases} 
\pi \left( i + A - \sum_{j \leq m} a_{\sigma(j)} \right) & \text{if } \sum_{j < m} k_j + \sum_{j \leq m} a_{\sigma(j)} < i \leq \sum_{j < m} k_j + \sum_{j \leq m} a_{\sigma(j)} \\
\pi \left( i - \sum_{j < m} k_j + \sum_{j \leq m} a_{\sigma(j)} \right) & \text{if } \sum_{j < m} k_j + \sum_{j \leq m} a_{\sigma(j)} < i \leq \sum_{j < m} k_j + \sum_{j \leq m} a_{\sigma(j)} 
\end{cases}
\]

Furthermore, given this notation,

\[
X_{\hat{\pi}} = x_{\hat{\pi}(1)}, \ldots, x_{\hat{\pi}(r)} \text{ and } \epsilon(\hat{\pi}) = \epsilon(\pi)(-1)^{\alpha_1} \text{ and } \chi(\hat{\pi}) = \chi(\pi)(-1)^{\alpha_2},
\]

where \( \alpha_1 = \sum_{i < j \in [n] \setminus \{\sigma(i) > \sigma(j)\}} |X_{\sigma(i)}| |X_{\sigma(j)}| + \sum_{i=1}^{n} |X_{\sigma(i)}| |x_{\pi(1+A)} + \cdots + x_{\pi(k_0+\cdots+k_{i-1}+A)}| \)

and \( \alpha_2 = \alpha_1 + \sum_{i < j \in [n] \setminus \{\sigma(i) > \sigma(j)\}} a_{\sigma(i)} a_{\sigma(j)} + \sum_{i < j} a_{\sigma(i)} k_j. \)

Proof. Careful examination of the definition of \( \hat{\pi} \) reveals that the first formula moves “free” strings of the form \( x_{\pi(1+k_0+\cdots+k_{i-1})}, \ldots, x_{\pi(k_0+\cdots+k_i)} \) into place (for \( 0 \leq m \leq n \)), and the second formula relocates the strings \( X_{\sigma(i)} \) (for \( 1 \leq m \leq n \)). Thus \( X_{\hat{\pi}} = x_{\hat{\pi}(1)}, \ldots, x_{\hat{\pi}(r)} \).

Furthermore, when \( x_{\pi(1)}, \ldots, x_{\pi(r)} \) are permuted to yield \( x_{\hat{\pi}(1)}, \ldots, x_{\hat{\pi}(r)} \), the Koszul sign is \((-1)^{\alpha_1}\), where the first sum in \( \alpha_1 \) comes from \( \sigma \) permuting the \( X_i \) strings, and the second sum comes from moving the “free” strings into place.
Finally, the additional sums in $\alpha$ count the transpositions, yielding the correct antisymmetric Koszul sign.

\[ η \sum_{i<j} w_{\sigma(i)} w_{\sigma(j)} + \sum_{i>j} w_{\sigma(i)} w_{\sigma(j)} + \sum_{i>j} v_{\sigma(i)} w_{\sigma(j)} = 0 \mod 2. \]

\[ η \sum_{i<j} v_{\sigma(i)} w_{\sigma(j)} + \sum_{i>j} v_{\sigma(i)} w_{\sigma(j)} = \sum_{i<j} (i-1)v_i + \sum_{i>j} (i-1)v_{\sigma(j)} \mod 2. \]

**Proof.** To prove the first assertion, we note that

\[ η \sum_{i<j} w_{\sigma(i)} w_{\sigma(j)} + \sum_{i>j} w_{\sigma(i)} w_{\sigma(j)} + \sum_{i>j} v_{\sigma(i)} w_{\sigma(j)} = 0 \mod 2, \]

which is congruent ($\mod 2$) to $\sum_{i<j} v_{\sigma(i)} w_{\sigma(j)} + \sum_{i>j} v_{\sigma(i)} w_{\sigma(j)} = \sum_{i>j} v_i w_j$. We denote this expression by $\sum_{i<j} v_{\sigma(i)} + \sum_{i>j} v_{\sigma(i)}$, (by the first assertion). Since all $w$-terms are odd, this is congruent to

\[ η \sum_{j=1}^n \sum_{i=j+1}^n (v_i + v_{\sigma(i)}) = \sum_{j=1}^n (j-1)v_j + \sum_{j=1}^n (j-1)v_{\sigma(j)}. \]

\[ \sum_{i<j} \sum_{\sigma(i)>\sigma(j)} v_{\sigma(i)} w_{\sigma(j)} + \sum_{i>j} \sum_{\sigma(i)>\sigma(j)} v_{\sigma(i)} w_{\sigma(j)} = \sum_{i<j} (i-1)v_i + \sum_{i>j} (i-1)v_{\sigma(j)} \mod 2. \]

5. **Symmetrization of Brace Algebras**

Given a (non-symmetric) brace structure $\{,\}$ on a graded vector space, we can define a symmetric brace structure $\langle,\rangle$ via

\[ f\langle g_1, \ldots, g_n \rangle := \sum_{\sigma \in S_n} \epsilon(\sigma) f\{g_{\sigma(1)}, \ldots, g_{\sigma(n)}\}. \]

Clearly, this satisfies the first symmetric brace axiom, since $f\langle \rangle = f\{\} = f$. We show in Theorem (15) that it satisfies the second symmetric brace axiom given in Definition (5), so this does in fact induce a symmetric brace structure. First, however, we need the following two lemmas, which are analogous to Lemmas (8) and (9).

\[ \sum_{\rho \in S_{n+m}} \epsilon(\rho) f\{x_{\rho(1)}, \ldots, x_{\rho(n)}\} = f_n \circ \theta_m(x_1, \ldots, x_{n+m}), \]

where

\[ \Theta_m(y_1, \ldots, y_n, z_1, \ldots, z_m) = \sum_{\pi \in S_m} \epsilon(\pi)(y_1, \ldots, y_n, z_{\pi(1)}, \ldots, z_{\pi(m)}) \] and

\[ \hat{f}_n(y_1, \ldots, y_n, z_1, \ldots, z_m) = \sum_{\sigma \in S_n} \epsilon(\sigma) \sum_{k_0 + \cdots + k_{n-1} = m} (-1)^n f\{z_1, \ldots, z_{k_0}, y_{\sigma(1)}, z_{1+k_0}, \ldots, y_{\sigma(n)}, z_{1+k_0+\cdots+k_{n-1}}, \ldots, z_m\}, \]

with a Koszul sign given by $\eta = \sum_{i=1}^n y_{\sigma(i)} [z_1 + \cdots + z_{k_0+k_1+\cdots+k_{i-1}}].$
Lemma 13. If $N = a_1 + \cdots + a_n$, then $\sum_{\pi \in S_N} \epsilon(\pi) (x_{\pi(1)}, \ldots, x_{\pi(N)})$ is equal to

$$\sum_{\gamma \in \{a_1, \ldots, a_n\}} \epsilon(\gamma) \sum_{\pi \in S_n} \epsilon(\pi_1) \cdots \epsilon(\pi_n) x_{\gamma(\pi(1))} \cdots x_{\gamma(\pi(n))} x_{\gamma(\pi(1)+a_1)} \cdots x_{\gamma(\pi(n)+\sum_{i=1}^{n-1} a_i)}.$$ 

Remark 14. Although a brace structure allows operators $g$ which accept an arbitrary number of inputs, it will be convenient in the proof of the following theorem to let $g^a$ denote the restriction of $g$ which accepts only exactly $a$ inputs.

Theorem 15. Given a (non-symmetric) brace structure $\{,\}$ on a graded vector space, define $\langle \cdot, \cdot \rangle$ via

$$f(g_1, \ldots, g_n) := \sum_{\sigma \in S_n} \epsilon(\sigma) f(g_{\sigma(1)}, \ldots, g_{\sigma(n)}).$$

Then $\langle g_1, \ldots, g_n \rangle (x_1, \ldots, x_r)$ is equal to

$$\sum_{\gamma \text{ is } (n+1)\text{-unshuffle}} \epsilon \cdot f(\langle g_1 x_{\gamma(1)} \cdots x_{\gamma(a_1)} \rangle, \ldots, g_n x_{\gamma(1)+\sum_{i=1}^{n-1} a_i} \cdots x_{\gamma(n)+\sum_{i=1}^{n-1} a_i}), x_{\gamma(r)}),$$

where $\epsilon$ is the Koszul sign of the permutation which maps $(g_1, \ldots, g_n, x_1, \ldots, x_r)$ to

$$(g_1, x_{\gamma(1)}, \ldots, x_{\gamma(a_1)}, g_2, \ldots, x_{\gamma(1)+\sum_{i=1}^{n-1} a_i}, \ldots, x_{\gamma(n)+\sum_{i=1}^{n-1} a_i}, g_n, x_{\gamma(r)}).$$

Proof. First, we will look at the right hand side.

If we temporarily denote

$$h_k = g_k \langle x_{\gamma(1)+\cdots+a_{k-1}}, \ldots, x_{\gamma(a_1+\cdots+a_k)} \rangle$$

$$= \sum_{\pi_k \in S_{a_k}} \epsilon(\pi_k) g_k \{x_{\gamma(\pi_k(1)+a_1+\cdots+a_{k-1})}, \ldots, x_{\gamma(\pi_k(a_k)+a_1+\cdots+a_{k-1})}\},$$

and denote $A = \sum_{i=1}^n a_i$, then the right hand side is equal to

$$\sum_{a_1+\cdots+a_{n+1}=r \& \gamma \text{ is } (a_1|\ldots|a_{n+1}) \text{ unshuffle}} (-1)^\nu \epsilon(\gamma) f(h_1, \ldots, h_n, x_{\gamma(1)+A}, \ldots, x_{\gamma(a_{n+1}+A)}),$$

where $\nu = \sum_{i=2}^n g_i [x_{\gamma(1)} + \cdots + x_{\gamma(a_1+\cdots+a_{i-1})}]$ is a Koszul sign. After applying Lemma (12), this is equal to

$$\sum_{a_1+\cdots+a_{n+1}=r, \gamma \text{ is unshuffle}} (-1)^\nu \epsilon(\gamma) \tilde{f}_n \left(\sum_{\pi_{n+1} \in S_{a_{n+1}}} \epsilon(\pi_{n+1}) (h_1, \ldots, h_n, x_{\gamma(\pi_{n+1}(1)+A)}, \ldots, x_{\gamma(\pi_{n+1}(a_{n+1})+A)})\right),$$

where $\tilde{f}_n$ is as defined in Lemma (12). Now, we will pull all of the $x$ terms back out, in order to apply Lemma (13). Note that the Koszul signs from this transformation merely cancel out $(-1)^\nu$. We then have the following long formula:
Now, though, we can apply Lemma (13), which yields the much shorter formula,
\[ \sum_{(a_i, \gamma)} \epsilon(\gamma) \sum_{\pi_1 \in S_{a_1}} \cdots \sum_{\pi_{a_n+1} \in S_{a_{a_n+1}}} \epsilon(\pi_{a_n+1}) \tilde{f}_n(g_1^{\alpha_1}, \ldots, g_n^{\alpha_n}, 1^{\alpha_{n+1}}) \left( x_{\pi_1(1)} \cdots, x_{\pi_{a_1}(1)}, \ldots, x_{\pi_{a_n}(n)} \right) \]
Before continuing, we need to pull all of the \( x \) terms back inside. In order to make our expressions a bit shorter, let \( X_i \) denote the input to \( g_i \). In other words, define
\[ X_i = x_{\pi(a_1 + \cdots + a_i)} \text{ for } i \in \{1 \ldots n\}. \]
It will also be convenient to let \( |X_i| \) denote the sum of the degrees of the variables in \( X_i \). When we pull the \( x \)-terms inside and use the more concise notation just defined, the formula for the right hand side becomes
\[ \sum_{(a_i)} \sum_{\pi \in S_r} \epsilon(\pi) (-1)^{\tilde{\nu}} \tilde{f}_n(g_1(X_1), \ldots, g_n(X_n), x_{\pi(1+A)}, \ldots, x_{\pi(r)}), \]
where \( \tilde{\nu} = \sum_{j<i} g_i |X_j| \). After expanding \( \tilde{f}_n \), the right hand side is equal to
\[ \sum_{(a_i)} \sum_{\pi \in S_r} \epsilon(\pi) \sum_{\sigma \in S_{a_1}} \epsilon(\sigma) \sum_{k_0 + \cdots + k_{a_n+1} = a_{n+1}} (-1)^{\eta} f \left( x_{\pi(1+A)} + \cdots + x_{\pi(k_0+A)}, g_{\sigma(1)}(X_{\sigma(1)}), \ldots, x_{\pi(r)} \right). \]
Here, \( \eta = \sum_{i=1}^n (g_{\sigma(i)} + |X_{\sigma(i)}|) \left( x_{\pi(1+A)} + \cdots + x_{\pi(k_0+A) + \cdots + k_{i-1}+A} \right) \)
and \( \epsilon(\sigma) = (-1)^{\lambda} \), where \( \lambda = \sum_{i<j \text{ or } \sigma(i) > \sigma(j)} (g_{\sigma(i)} + |X_{\sigma(i)}|) (g_{\sigma(j)} + |X_{\sigma(j)}|) \).
Now, we will look at the left hand side. \( f(g_1, \ldots, g_n) \langle x_1, \ldots, x_r \rangle \) is equal to
\[ \sum_{\sigma \in S_n} \epsilon(\sigma) \left\{ g_{\sigma(1)}, \ldots, g_{\sigma(n)} \right\} \langle x_1, \ldots, x_r \rangle, \]
which is equal to
\[ \sum_{\sigma \in S_n} \epsilon(\sigma) \sum_{\pi \in S_r} \epsilon(\pi) f \left( g_{\sigma(1)}, \ldots, g_{\sigma(n)} \right) \{x_{\pi(1)}, \ldots, x_{\pi(r)}\}. \]
If we apply Definition (1) and let \( g_i^{a_i} \) denote the restriction of \( g_i \) which accepts exactly \( a_i \) inputs, then the left hand side is equal to
\[ \sum_{\sigma \in S_n} \epsilon(\sigma) \sum_{\pi \in S_r} \epsilon(\pi) \sum_{k_0 + \cdots + k_{a_{n+1}} = a_{n+1}} (-1)^{\alpha} f \left( 1^{k_0}, g_{\sigma(1)}^{a_{\sigma(1)}}, \ldots, g_{\sigma(n)}^{a_{\sigma(n)}}, 1^{k_{a_n}} \right) \{x_{\pi(1)}, \ldots, x_{\pi(r)}\}. \]
After applying Lemma (10), this is equal to
\[ \sum_{\sigma \in S_n} \sum_{(k_i, a_i)} \sum_{\pi \in S_r} \epsilon(\pi) (-1)^{\alpha} f \left( 1^{k_0}, g_{\sigma(1)}^{a_{\sigma(1)}}, \ldots, g_{\sigma(n)}^{a_{\sigma(n)}}, 1^{k_{a_n}} \right) \{x_{\pi(1+A)}, \ldots, x_{\pi(k_0+A)}, X_{\sigma(1)}, \ldots, X_{\sigma(n)}\} \]
\[ \{ x_{\pi(1+k_0+A)}, \ldots, x_{\pi(r)} \}. \]
where $\alpha_1$ is given in Lemma (10). Finally, when the $x$-terms are moved inside, the left hand side is equal to

$$\sum \epsilon(\sigma) \sum_{\langle a_1, a \rangle} \sum_{\pi \in S_r} \epsilon(\pi)(-1)^{\alpha_1 + \mu} f \{ x_{\pi(1+A)}, \ldots, x_{\pi(k_0+A)} \}, g_{\sigma(1)}(X_{\sigma(1)}), \ldots, g_{\sigma(n)}(X_{\sigma(n)}), x_{\pi(1+k_0+A)}, \ldots, x_{\pi(r)} \}.$$  

Here, $\mu = \sum_i g_{\sigma(i)}[x_{\pi(1+A)} + \cdots + x_{\pi(k_0+\cdots+k_{i-1}+A)}] + \sum_{j<i} g_{\sigma(j)}|X_{\sigma(i)}|$, and $\epsilon(\sigma) = (-1)^{\zeta}$, where $\zeta = \sum_{i<j \land \sigma(i) > \sigma(j)} g_{\sigma(i)}g_{\sigma(j)}$.

Now that the terms on both sides are easy to compare, it is clear that the two sides are equal if and only if $\bar{\nu} + \chi + \lambda + \eta + \zeta + \alpha_1 + \mu \equiv 0 \pmod{2}$.

After making the most obvious cancellations, we see that $\bar{\nu} + \chi + \lambda + \eta + \zeta + \alpha_1 + \mu$ is congruent to

$$\sum_{j<i} g_i x_j + \sum_{i<j \land \sigma(i) > \sigma(j)} (g_{\sigma(j)}|X_{\sigma(j)}| + g_{\sigma(j)}|X_{\sigma(i)}|) + \sum_{j<i} g_{\sigma(j)}|X_{\sigma(i)}|,$$

which is congruent to zero (mod 2) by Lemma (11). \hfill \Box

6. Symmetrization of the Brace Structure on $\bigoplus_{k \geq 1} \text{Hom}(V^\otimes k, V)$

In this section, we will demonstrate a nice relationship between the brace defined in Example 2 and the symmetric brace defined in Example 6, by showing that the symmetrization of the non-symmetric brace structure on $\text{Hom}(V^\otimes k, V)$ is equal to the symmetric brace of the anti-symmetrized maps. Specifically, we have

**Theorem 16.** $\sum_{\sigma \in S_n} \epsilon(\sigma) a_{\sigma(f, g_1, \ldots, g_n)} = a_{\sigma(f)}(a_{g_1}, \ldots, a_{g_n})$.

**Proof.** First, we will manipulate the right hand side. Using the symmetric brace structure defined in Example 6, $a_{\sigma(f)}(a_{g_1}, \ldots, a_{g_n})$ is equal to

$$(-1)^d \sum_{\gamma \text{ is an unshuffle}} \chi(\gamma) a_{\sigma(f)}(a_{g_1} \otimes \cdots \otimes a_{g_n} \otimes 1^{N-n}) x_{\gamma(1)}, \ldots, x_{\gamma(r)}$$

where $\delta$ is given in Example 6.

When we substitute the $x$ terms using the Koszul convention and suppress the tensor notation, this is equal to

$$(-1)^\nu \sum_{\gamma} \chi(\gamma)(-1)^\nu a_{\sigma(f)}(h_1, \ldots, h_n, x_{\gamma(1)} + \cdots + x_{\gamma(a_1+\cdots+a_{i-1})})$$

where $\nu = \sum_{i=2}^n q_i [x_{\gamma(1)} + \cdots + x_{\gamma(a_1+\cdots+a_{i-1})}]$.

and $h_k = a_{\sigma(g_k)}(x_{\gamma(1+a_1+\cdots+a_{k-1})}, \ldots, x_{\gamma(a_1+\cdots+a_k)})$

$$= \sum_{\pi_k \in S_{a_k}} \chi(\pi_k) g_k (x_{\gamma(\pi_k(1+a_1+\cdots+a_{k-1}))}, \ldots, x_{\gamma(\pi_k(a_k)+a_1+\cdots+a_{k-1})})$$.
If we denote \( A = \sum_{i=1}^{n} a_i \) and apply Lemma (8), this is equal to

\[
\sum_{\gamma} \chi(\gamma)(-1)^{\delta + \nu} f \circ \Phi_{n} \circ \Psi_n \left( \sum_{\pi_{an+1} \in S_{an+1}} \chi(\pi_{an+1}) (h_1, \ldots, h_m, x_\gamma(\pi_{an+1}(1)+A), \ldots, x_\gamma(\pi_{an+1}(a_n+1)+A)) \right).
\]

Now, we will pull all of the \( x \) terms back out, in order to apply Lemma (9). Note that the Koszul signs from this transformation merely cancel out \((-1)^\nu\). We then have the following long formula, which spans two lines!

\[
(-1)^\delta \sum_{\gamma} \chi(\gamma) \sum_{\pi \in S_r} \chi(\pi) \cdot \sum_{\pi_{an+1} \in S_{an+1}} \chi(\pi_{an+1}) f \circ \Phi_{na} \circ \Psi_n (g_1, \ldots, g_n, 1^{an+1})
\]

\[
\left( x_\gamma(\pi(1)), \ldots, x_\gamma(\pi(a_1)), x_\gamma(\pi(2)+a_1), \ldots, x_\gamma(\pi(a_n), x_\gamma(\pi(a_n+1)+1), \ldots, x_\gamma(\pi(a_n+1)+A) \right).
\]

Now, though, we can apply Lemma (9), which yields the much shorter formula,

\[
(-1)^\delta \sum_{\pi \in S_r} \chi(\pi) f \circ \Phi_{na} \circ \Psi_n (g_1, \ldots, g_n, 1^{an+1}) \left( x_{\pi(1)}, \ldots, x_{\pi(r)} \right).
\]

Before continuing, we need to pull all of the \( x \) terms back inside. In order to make our expressions a bit shorter, let \( X_i \) denote the input to \( g_i \), and let \( X_{n+1} \) denote the free \( x \) terms (letting \( a_{n+1} = N-n \)). In other words, define

\[
X_i = x_{\pi(1+a_1+\cdots+a_{i-1})}, \ldots, x_{\pi(a_1+\cdots+a_i)}.
\]

It will also be convenient to let \( |X_i| \) denote the sum of the degrees of the variables in \( X_i \). When we pull the \( x \)-terms inside and use the more concise notation just defined, the formula for the right hand side becomes

\[
(-1)^\delta \sum_{\pi \in S_r} \chi(\pi)(-1)^\tilde{\nu} f \circ \Phi_{n,N-n} \circ \Psi_n (g_1(X_1), \ldots, g_n(X_n), X_{n+1}),
\]

where \( \tilde{\nu} = \sum_{j<i} q_i |X_j| \). After expanding \( \Psi_n \), the right hand side is equal to

\[
(-1)^\delta \sum_{\pi \in S_r} \chi(\pi)(-1)^\tilde{\nu} f \circ \Phi_{n,N-n} \left( \sum_{\sigma \in S_n} \chi(\sigma) \left( g_{\sigma(1)}(X_{\sigma(1)}), \ldots, g_{\sigma(n)}(X_{\sigma(n)}), X_{n+1} \right) \right).
\]

In the above expression, \( \chi(\sigma) \) is equal to \((-1)^\lambda\), where

\[
\lambda = \sum_{\substack{i<j \leq n, \\ \sigma(i) > \sigma(j)}} \left[ \left( q_{\sigma(i)} + |X_{\sigma(i)}| \right) \left( q_{\sigma(j)} + |X_{\sigma(j)}| \right) + 1 \right].
\]

Now, if we expand \( \Phi_{n,N-n} \), we get

\[
\sum_{\pi \in S_r, \sigma \in S_n, \ k_0+\cdots+k_n = N-n} \chi(\pi)(-1)^{\delta + \tilde{\nu} + \lambda + \eta} f \left( x_{\pi(1+A)}, \ldots, x_{\pi(k_0+A)}, g_{\sigma(1)}(X_{\sigma(1)}), x_{\pi(1+k_0+A)}, \ldots, g_{\sigma(n)}(X_{\sigma(n)}), x_{\pi(1+k_0+\cdots+k_n+1+A)}, \ldots, x_{\pi(r)} \right),
\]

where \( \eta = \sum_{i=1}^{n} \left\{ (q_{\sigma(i)} + |X_{\sigma(i)}|) \left( x_{\pi(1+A)} + \cdots + x_{\pi(k_0+\cdots+k_{i-1}+A)} \right) + (n-i)k_i \right\}. \)
Now, we will work with the left hand side of the equation. Using the brace defined in Example 2, 
\[ \sum_{\sigma \in S_n} \epsilon(\sigma) \Delta \{ g_{\sigma(1)}, \ldots, g_{\sigma(n)} \} (x_1, \ldots, x_r) \] 
is equal to 
\[ \sum_{\sigma \in S_n} \epsilon(\sigma) \Delta \left( \sum_{k_0 + \ldots + k_n = N-n} (-1)^{\beta} f(1^{\otimes k_0} \otimes g_{\sigma(1)} \otimes 1^{\otimes k_1} \otimes \ldots \otimes 1^{\otimes k_{n-1}} \otimes g_{\sigma(n)} \otimes 1^{\otimes k_n}) \right) (x_1, \ldots, x_r), \]
where \( \beta \) is given in Example 2. Note also that the Koszul sign \( \epsilon(\sigma) \) must be calculated using the degree of \( g_i \) as an element of the symmetric brace algebra (so \( |g_i| = q_i + a_i - 1 \)). Thus \( \epsilon(\sigma) = (-1)^{\zeta} \), where 
\[ \zeta = \sum_{i<j \& \sigma(i) > \sigma(j)} (q_{\sigma(i)} + a_{\sigma(i)} - 1)(q_{\sigma(j)} + a_{\sigma(j)} - 1). \]

If we now antisymmetrize by taking all signed permutations of the \( x \)'s, and suppress the tensor notation, this is equal to 
\[ \sum_{\pi \in S_r} \sum_{\sigma \in S_n} (-1)^{\beta + \zeta} f(1^{k_0}, g_{\sigma(1)}, 1^{k_1}, \ldots, 1^{k_{n-1}}, g_{\sigma(n)}, 1^{k_n}) \left( \sum_{\pi \in S_r} \chi(\pi) \left( x_{\pi(1)}, \ldots, x_{\pi(r)} \right) \right). \]

After applying Lemma (10), the left hand side is equal to 
\[ \sum_{\pi \in S_r, \sigma \in S_n, k_0 + \ldots + k_n = N-n} (-1)^{\beta + \zeta + \alpha_2} \chi(\pi) f(1^{k_0}, g_{\sigma(1)}, 1^{k_1}, \ldots, g_{\sigma(n)}, 1^{k_n}) \left( x_{\pi(1) + A}, \ldots, x_{\pi(k_0 + A)}, X_{\sigma(1)}, \ldots, x_{\pi(k_0 + \ldots + k_n + A)}, \ldots, x_{\pi(r)} \right), \]
where \( \alpha_2 \) is given in Lemma (10).

Finally, when the variables are moved inside, the left hand side is equal to 
\[ \sum_{\pi \in S_r, \sigma \in S_n, k_0 + \ldots + k_n = N-n} (-1)^{\beta + \zeta + \alpha_2 + \mu} \chi(\pi) f(1^{k_0}, g_{\sigma(1)}, X_{\sigma(1)}, x_{\pi(1) + A}, \ldots, x_{\pi(k_0 + A)}, \ldots, g_{\sigma(n)}(X_{\sigma(n)}), x_{\pi(k_0 + \ldots + k_n + A)}, \ldots, x_{\pi(r)}), \]
where \( \mu = \sum_i q_{\sigma(i)} \left| x_{\pi(1) + A} + \cdots + x_{\pi(k_0 + \ldots + k_{i-1} + A)} \right| + \sum_{j<i} q_{\sigma(j)} \left| X_{\sigma(j)} \right|. \)

Since the right hand side is equal to 
\[ \sum_{\pi \in S_r, \sigma \in S_n, k_0 + \ldots + k_n = N-n} \chi(\pi)(-1)^{\beta + \zeta + \alpha_2 + \mu + \delta + \nu + \lambda + \eta} \left( x_{\pi(1) + A}, \ldots, x_{\pi(k_0 + A)}, g_{\sigma(1)}(X_{\sigma(1)}), x_{\pi(1) + k_0 + A}, \ldots, g_{\sigma(n)}(X_{\sigma(n)}), x_{\pi(k_0 + \ldots + k_n + A)}, \ldots, x_{\pi(r)} \right), \]
we see that the two sides are equal if and only if 
\[ \beta + \zeta + \alpha_2 + \mu + \delta + \nu + \lambda + \eta \equiv 0 \ (mod \ 2). \]

After cancelling the most obvious terms, \( \beta + \zeta + \alpha_2 + \mu + \delta + \nu + \lambda + \eta \) is congruent to
\[
\sum_i (n-i)a_{\sigma(i)} + \sum_i (N-i)q_{\sigma(i)} + \sum_j q_{\sigma(i)}a_{\sigma(j)} \\
+ \sum_{i<j \& \sigma(i)\sigma(j)} q_{\sigma(i)}a_{\sigma(j)} + q_{\sigma(i)} + a_{\sigma(i)}q_{\sigma(j)} + a_{\sigma(i)} + q_{\sigma(j)} + a_{\sigma(j)} + \sum_{j<i} q_{\sigma(i)}|X_{\sigma(j)}| \\
+ \sum_i (N-i)q_i + \sum_j q_i a_j + \sum_i (n-i)a_i + \sum_j q_j |X_j| + \sum_{i<j \& \sigma(i)\sigma(j)} [q_{\sigma(i)}|X_{\sigma(j)}| + |X_{\sigma(i)}|q_{\sigma(j)}].
\]

After applying Lemma (11), this is congruent to

\[
\sum_i \left\{(n-i)a_{\sigma(i)} + (N-i)q_{\sigma(i)} + (i-1) \left[a_i + a_{\sigma(i)} + q_i + q_{\sigma(i)} \right] + (N-i)q_i + (n-i)a_i \right\},
\]

which is equal to \( \sum_i \left\{(n-1)\left[a_{\sigma(i)} + a_i \right] + (N-1)\left[q_{\sigma(i)} + q_i \right] \right\} \equiv 0 \pmod{2}. \)

As a corollary, we obtain Theorem 3.1 of [4]:

**Corollary 17.** The anti-symmetrization \( l := as(\mu) \) of an \( A_\infty \)-algebra structure \( \mu \) yields an \( L_\infty \)-algebra structure.

**Proof.** Given \( \mu\{\mu\} = 0 \) (recall Remarks (3) and (7)) we have

\[0 = as(\mu\{\mu\}) = as(\mu)\langle as(\mu) \rangle = l(l).\]

\[\square\]

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**References**


