

Moduli spaces of L_∞ structures

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Introductory L_∞ papers:

- Lada & Stasheff, *Introduction to sh Lie algebras for physicists*. Int. J. Theo. Phys. 32, 1087-1103 (1993).
- Lada & Markl, *Strongly homotopy Lie algebras*. Communications in Algebra, 23, 2147-2161 (1995).

L_∞ structure: Generalized Lie algebra.

Graded Lie algebra: skew bilinear bracket $[,] : V \otimes V \rightarrow V$ of degree 0, satisfying Jacobi identity.

L_∞ structure: skew n -linear “brackets” $l_n : V^{\otimes n} \rightarrow V$ of degree $2 - n$, satisfying an infinite number of generalized Jacobi identities, $\sum_{p=1}^n (-1)^{p(n-p)} l_{n-p+1} \circ l_p = 0$.

Generalized Jacobi identities:

n=1: $l_1 \circ l_1 = 0$. “ l_1 is a differential.”

n=2: $-l_2 \circ l_1 + l_1 \circ l_2 = 0$.

“ l_2 is a chain map.”

n=3: $l_3 \circ l_1 + l_2 \circ l_2 + l_1 \circ l_3 = 0$.

“ $l_2 \circ l_2$ is *homotopic* to zero.”

n>3: : “Higher homotopies.”

L_∞ structures as codifferentials:

V : \mathbf{Z}_2 or \mathbf{Z} -graded vector space.

Define W by $W_i = V_{i+1}$ (desuspension).

Then we can define L_∞ structures as codifferentials on the (graded) symmetric coalgebra of W , $S(W)$.

\mathbf{Z} -grading: Degree 1 codifferentials.

\mathbf{Z}_2 -grading: Odd codifferentials.

Moduli space of L_∞ structures:

Let d, d' be codifferentials on W, W' .

L_∞ morphism: Coalgebra morphism

$g : S(W') \rightarrow S(W)$ satisfying $gd' = dg$.

If $W = W'$ and g is an automorphism of $S(W)$,
then $d' = g^{-1}dg$. ($d' \sim d$.)

Thus $\text{Aut}(S(W))$ acts on L_∞ structures, and we
denote $g^*(d) = g^{-1}dg$. The equivalence classes of
this action form the *moduli space* of L_∞ algebras.

L_∞ morphisms:

\mathbf{Z} -grading: A morphism of symmetric algebras must be a degree 0 map.

\mathbf{Z}_2 -grading: The map must be even.

Therefore, every morphism of **\mathbf{Z} -graded** symmetric algebras is also a morphism of **\mathbf{Z}_2 -graded** symmetric algebras.

Any two equivalent **\mathbf{Z} -graded** L_∞ structures are also equivalent as **\mathbf{Z}_2 -graded** L_∞ structures!

Moduli space of L_∞ structures, cont'd:

Thus there is a map from the moduli space of \mathbf{Z} -graded L_∞ structures to the moduli space of \mathbf{Z}_2 -graded L_∞ structures.

There are fewer \mathbf{Z} -graded L_∞ structures on W ...

But the notion of equivalence is more restrictive.

Thus it is not obvious whether the map between these moduli spaces is injective or surjective. We have found that in general it is neither.

Deformation theory:

If d_k is a codifferential, and $\{\delta_i\}$ is a complete set of representatives for the cohomology, then the *universal infinitesimal deformation* d^∞ of d_k is $d^\infty = d_k + \delta_i t^i$, where (t^i) are parameters.

The *miniversal deformation* is the most general extension of the universal infinitesimal deformation to a formal deformation, and its primary importance is that it induces all possible formal deformations.

Deformation theory, cont'd

To construct the miniversal deformation, we compute $[d_\infty, d_\infty]$, and look at the higher order terms which appear.

Since the bracket must be a cocycle, it can be expressed as a sum of coboundary terms and nontrivial cocycles.

Since coboundaries may be eliminated by adding higher order terms, we can get a coderivation d_∞ whose self-bracket contains no coboundaries.

Deformation theory, cont'd

However, the nontrivial cocycles appearing in the bracket cannot be eliminated by this process.

Accordingly, we obtain some relations on the parameters which must be satisfied if the bracket is to vanish.

These relations are called the relations on the base of the versal deformation, which is the algebra $\mathbf{k}[[t^1, t^2, \dots]]$ modulo the ideal generated by these relations.

Notation:

$C^r = \text{Hom}(S^r(W), W)$ is the space of r -cochains.

$C_s^r = r$ -cochains of internal degree s .

A \mathbf{Z} -graded codifferential d lies in $C_1^\bullet = \prod_{r=1}^{\infty} C_1^r$,

so $d = d_1 + d_2 + \cdots$, where $d_r \in C_1^r$.

There is a natural isomorphism between

$C = \prod_{r=1}^{\infty} C^r$ and $\text{Coder}(S(W))$, so that C

inherits the structure of a graded Lie algebra.

Moreover, $[C_m^k, C_n^l] \subseteq C_{m+n}^{k+l-1}$.

When d is a codifferential, $[d, d] = 0$.

Equivalently: $\sum_{r+s=n+1} [d_r, d_s] = 0 \quad \forall n \in N$.

If k is the least integer such that $d_k \neq 0$, d_k is itself a codifferential (and d is an *extension* of d_k).

If we define $D : C \rightarrow C$ by $D(\phi) = [d_k, \phi]$, then D is a codifferential on C ; i.e., $D^2 = 0$.

This allows us to define the cohomology $H(d_k)$ as $H(d_k) = Ker(D)/Im(D)$, where $H^n(d_k) = Ker(D : C^n \rightarrow C^{n+k-1})/Im(D : C^{n-k+1} \rightarrow C^n)$.

L_∞ structures as deformations:

We can re-express the relations in the form

$$D(d_{n+1}) = -\frac{1}{2} \sum_{r=k+1}^n [d_r, d_{k+n+1-r}] \quad \forall n \geq k.$$

Essentially, this implies that L_∞ algebras are deformations of the codifferential of the leading term.

An automorphism λ of $S(W)$ is called *linear* if $\lambda(w_1 \cdots w_n) = \lambda(w_1) \cdots \lambda(w_n)$.

Not every automorphism is linear, but every automorphism g can be uniquely expressed in the form $g = \lambda \prod_{n=2}^{\infty} \exp(\phi_n)$, where λ is linear and $\phi_n \in C^n$.

We can express the action of g on codifferentials by the formula $g^* = (\prod_{n=2}^{\infty} \exp(-ad_{\phi_n}))\lambda^*$, so the action of the nonlinear part of g on C can be computed in terms of brackets of coderivations.

In order to express codifferentials concisely, we use the following notation.

Let $\{e_1, e_2, e_3\}$ be a fixed basis of W . Then $\phi_k^{i_1 i_2 i_3} \in \text{Hom}(S^{i_1+i_2+i_3}(W), W)$ is defined by

$$\phi_k^{i_1 i_2 i_3}(e_1^{j_1} e_2^{j_2} e_3^{j_3}) = i_1! i_2! i_3! \delta_{j_1}^{i_1} \delta_{j_2}^{i_2} \delta_{j_3}^{i_3} e_k.$$

In order to make it easier to distinguish between even and odd coderivations, if ϕ_l^{ijk} is odd, we will denote it as ψ_l^{ijk} instead.

Convention: List even elements of a basis first.

Codifferentials on $W = W_{-1} \oplus W_0 \oplus W_1$:

The basis elements of $W = W_{-1} \oplus W_0 \oplus W_1$ have degrees $|e_1| = 0$, $|e_2| = -1$ and $|e_3| = 1$, and the space of cochains has this decomposition:

- $C_{-2}^r = \langle \phi_2^{r-1,0,1} \rangle$
- $C_{-1}^r = \langle \psi_1^{r-1,0,1}, \psi_2^{r,0,0}, \psi_2^{r-2,1,1} \rangle$
- $C_0^r = \langle \phi_1^{r,0,0}, \phi_1^{r-2,1,1}, \phi_2^{r-1,1,0}, \phi_3^{r-1,0,1} \rangle$
- $C_1^r = \langle \psi_1^{r-1,1,0}, \psi_3^{r,0,0}, \psi_3^{r-2,1,1} \rangle$
- $C_2^r = \langle \phi_3^{r-1,1,0} \rangle.$

Codifferentials on $\mathbf{W} = \mathbf{W}_{-1} \oplus \mathbf{W}_0 \oplus \mathbf{W}_1$:

Since a codifferential for a \mathbf{Z} -graded L_∞ algebra must have degree 1, the most general order k codifferential is $d = \psi_1^{k-1,1,0} a + \psi_3^{k,0,0} b + \psi_3^{k-2,1,1} c$.

However, one can show that any such codifferential is equivalent to one of the following:

- $d_k(\lambda : \mu) = \psi_1^{k-1,1,0} \lambda + \psi_3^{k-2,1,1} \mu$
- $d_k^* = \psi_3^{k,0,0}$
- $d_k^\# = \psi_1^{k-1,1,0} + \psi_3^{k,0,0} + \psi_3^{k-2,1,1} k$.

Codifferentials on $W = W_{-1} \oplus W_0 \oplus W_1$:

When we extend the order k codifferentials, we obtain the following complete list of all nonequivalent codifferentials on W .

- $d_k(\lambda : \mu) = \psi_1^{k-1,1,0} \lambda + \psi_3^{k-2,1,1} \mu$
- $d_k^* = \psi_3^{k,0,0}$
- $d_{k,l}(\alpha) = \psi_3^{k-2,1,1} + \psi_1^{l-1,1,0} + \psi_1^{2l-k-1,1,0} \alpha,$
 $k \leq l$
- $d_{k,l}^\# = \psi_1^{k-1,1,0} + \psi_3^{k-2,1,1} l + \psi_3^{l,0,0}, \quad k \leq l.$

Deformations of $d_k(\lambda : \mu)$ ($\lambda \neq \mathbf{0}$, $\frac{\mu}{\lambda} \notin \mathbf{Z}$):

The miniversal infinitesimal deformation is

$$\begin{aligned} d_k^\infty(\lambda : \mu) &= d_k(\lambda : \mu) + \psi_1^{m-1,1,0} s_m + \psi_3^{n-2,1,1} t_n \\ &= d_k(\lambda : \mu + t_n) + \sum_{n=1}^{k-1} d_n(s_n : t_n). \end{aligned}$$

Thus $d_k(\lambda : \mu)$ can be deformed as follows.

- Smoothly: to $d_k(\lambda : \mu)$
- Jump: to ‘lower’ $d_n(\alpha : \beta) \forall n < k$
- Jump: to ‘lower’ $d_{m,n}(\beta) \forall m < n \leq k$

Deformations of $d_k(\mathbf{0} : \mathbf{1})$:

The miniversal infinitesimal deformation is

$$d_k^\infty(\mathbf{0} : \mathbf{1}) = d_k(\mathbf{0} : \mathbf{1}) + \psi_1^{m-1,1,0} s_m + \psi_3^{n-2,1,1} t_n,$$

$$1 \leq m < \infty, 1 < n < k.$$

Thus $d_k(\lambda : \mu)$ can be deformed as follows.

- Smoothly: to $d_k(\lambda : \mu)$
- Jump: to ‘lower’ $d_n(\alpha : \beta) \forall n < k$
- Jump: to $d_{m,n}(\beta) \forall m < n$
- Smoothly: to $d_{n,k}(\beta) \forall n < k$

Deformations of $d_{k,l}(\alpha)$:

$d_{k,l}(\alpha)$ can be deformed as follows.

- Jump: to $d_{m,n}(\beta)$ for $m < n < k$
- Jump: to $d_{k,k+1}(\beta)$
- Jump: to $d_m(\alpha : \beta)$ where $m \leq k$
- Smoothly: to neighborhood of $d_{n,l}(\alpha)$ ($n < k$)
- Smoothly: to a $d_{n,m}(\infty)$ in a neighborhood of $\beta = \infty \forall m \leq k \leq n < l$

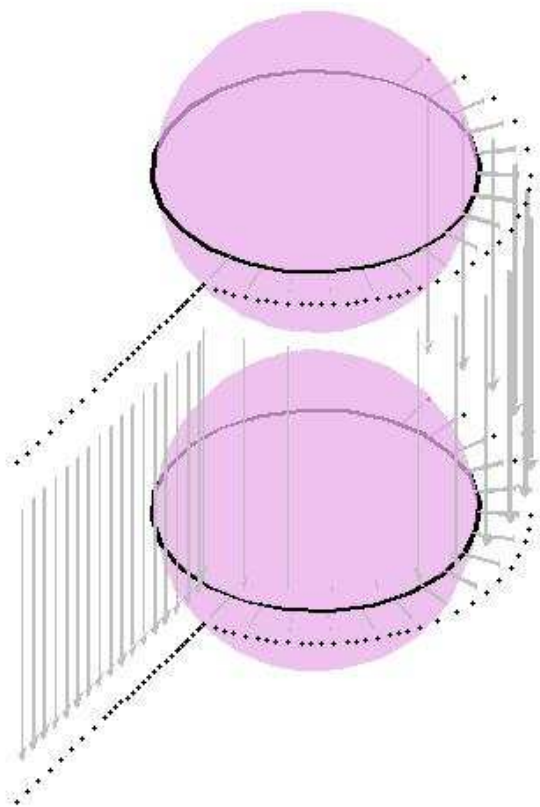
Deformations of d_k^* :

d_k^* can be deformed as follows.

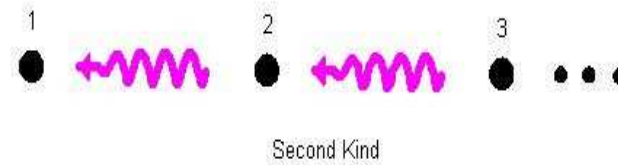
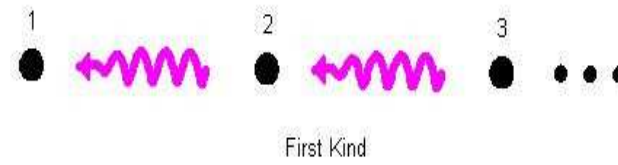
- Jump: to $d_{k,k}^\#$
- Jump: to d_m^* for $1 \leq m \leq k$
- Jump: to $d_{m,n}^\#$ where $1 \leq m < n < k$

Deformations of $d_{k,1}^\#$

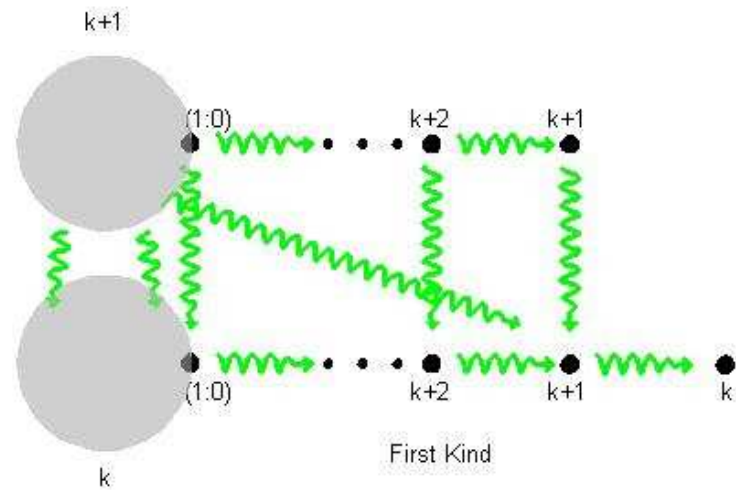
- Jump: to $d_{m,n}^\#$ where $m \leq k$
- Jump: to d_n^* for $n \leq k$



Moduli space of codifferentials on $\mathbf{W}_{-1} \oplus \mathbf{W}_0 \oplus \mathbf{W}_1$.



Moduli space of codifferentials on $\mathbf{W}_0 \oplus \mathbf{W}_1 \oplus \mathbf{W}_2$.



Moduli space of codifferentials on $\mathbf{W}_{-2} \oplus \mathbf{W}_{-1} \oplus \mathbf{W}_0$.