A Finite Dimensional A_∞ Algebra Example

Mike Allocca

NC State University

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A_∞ algebras

Definition

Let V be a graded vector space. An A_∞ algebra on V is a collection of linear maps $m_k : V^{\otimes k} \to V$ of degree $2-k$ that satisfy the identity

$$\sum_{\lambda=0}^{n-1} \sum_{k=1}^{n-\lambda} \alpha m_{n-k+1}(x_1 \otimes \cdots \otimes x_\lambda \otimes m_k(x_{\lambda+1} \otimes \cdots \otimes x_{\lambda+k}) \otimes x_{\lambda+k+1} \otimes \cdots \otimes x_n) = 0$$

where $\alpha = (-1)^{k+\lambda+k\lambda+kn+k(|x_1|+\cdots+|x_\lambda|)}$, for all $n \geq 1$.

- For $n=3$ in a 1-term graded vector space, this yields

$$m_3(m_1(x_1), x_2, x_3) + (-1)^{|x_1|} m_3(x_1, m_1(x_2), x_3) + (-1)^{|x_1||x_2|} m_3(x_1, x_2, m_1(x_3)) + m_2(m_2(x_1, x_2), x_3) - m_2(x_1, m_2(x_2, x_3)) + m_1(m_3(x_1, x_2, x_3)) = 0$$

- With m_2 acting as multiplication, this behaves like an associative algebra.

L_∞ algebras

Definition

Let V be a graded vector space. An L_∞ algebra on V is a collection of skew symmetric linear maps $l_n : V^{\otimes n} \to V$ of degree $2-n$ that satisfy the identity

$$\sum_{i+j=n+1} \sum_{\sigma} e(\sigma)(-1)^{\sigma}(-1)^{i(j-1)} l_j(l_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}) = 0$$

where $(-1)^{\sigma}$ is the sign of the permutation, $e(\sigma)$ is the sign that arises from the degrees of the permuted elements, and σ is taken over all $(i, n-i)$ unshuffles.

- For $n = 3$ in a 1-term graded vector space, this yields

$$l_3(l_1(x_1), x_2, x_3) - (-1)^{|x_1||x_2|} l_3(l_1(x_2), x_1, x_3) + (-1)^{|x_3|(|x_1|+|x_2|)} l_3(l_1(x_3), x_1, x_2)$$
$$+ l_2(l_2(x_1, x_2), x_3) + (-1)^{|x_2||x_3|} l_2(l_2(x_1, x_3), x_2))$$
$$+ (-1)^{|x_1|(|x_2|+|x_3|)} l_2(l_2(x_2, x_3), x_1) + l_1(l_3(x_1, x_2, x_3)) = 0$$

- With l_2 representing a Lie bracket, this behaves just like a Lie algebra.

Slick relationship

Recall: Given an associative algebra A with multiplication $\cdot$, A induces a Lie algebra through the commutator bracket: $[x, y] = x \cdot y - y \cdot x$

Similarly, an A_∞ algebra will induce an L_∞ algebra through its commutators:

- $l_1 := m_1$
- $l_2(x, y) := m_2(x, y) - (-1)^{|x||y|} m_2(y, x)$
- $l_3(x, y, z) := m_3(x, y, z) - (-1)^{|x||y|} m_3(y, x, z) - (-1)^{|y||z|} m_3(x, z, y) - (-1)^{|x||y|+|x||z|+|y||z|} m_3(z, y, x) + (-1)^{|z|(|x|+|y|)} m_3(z, x, y) + (-1)^{|x|(|y|+|z|)} m_3(y, z, x) = 0$
- etc..

Concrete examples are far from trivial

Problem: Due to the graded setting and infinite amount of maps, even construction of a simple ‘interesting’ example of an A_∞ or L_∞ algebra is difficult!

Marilyn Daily and Tom Lada constructed a finite dimensional L_∞ algebra example (2004).

A finite dimensional L_∞ algebra example

Example (Daily, Lada)

Consider the graded vector space $V = \oplus V_n$ where V_0 is a 2-dimensional space with basis $< v_1, v_2 >$ and V_1 is a 1-dimensional space with basis $< w >$. Let $V_n = 0$ for $n \neq 0, 1$. We define an L_∞ structure, $l_n : V^{\otimes n} \to V$, on V via the following maps:

$$l_1(v_1) = l_1(v_2) = w$$
$$l_2(v_1 \otimes v_2) = v_1, \quad l_2(v_1 \otimes w) = w$$
$$l_n(v_2 \otimes w^{\otimes n-1}) = C_n w \quad \text{for all } n \geq 3$$

where $C_n = (-1)^{\frac{(n-2)(n-3)}{2}}(n-3)!$. We extend these maps to be skew symmetric and define l_n to be 0 when evaluated on any element of $V^{\otimes n}$ that is not listed above.

What about a similar A_∞ algebra example?

Motivating question: Can we build an A_∞ algebra structure with the following properties?

1. Defined over the same graded vector space
2. Commutator induces Daily/Lada's L_∞ structure

Property 2 adds great difficulty in generating such a structure. For ease of computation, we only consider property 1.

A finite dimensional example

Theorem

Let V denote the graded vector space given by $V = \bigoplus V_n$ where $V_0 =< v_1, v_2 >$, $V_1 =< w >$, and $V_n = 0$ for $n \neq 0, 1$. Define a structure on V by the following linear maps $m_n : V^{\otimes n} \to V$:

$$m_1(v_1) = m_1(v_2) = w$$

$$\text{For } n \geq 2: \quad m_n(v_1 \otimes w^{\otimes k} \otimes v_1 \otimes w^{\otimes (n-2)-k}) = (-1)^k s_n v_1, \; 0 \leq k \leq n-2$$

$$m_n(v_1 \otimes w^{\otimes (n-2)} \otimes v_2) = s_{n+1} v_1$$

$$m_n(v_1 \otimes w^{\otimes (n-1)}) = s_{n+1} w$$

Where $s_n = (-1)^{\frac{(n+1)(n+2)}{2}}$, and $m_n = 0$ when evaluated on any element of $V^{\otimes n}$ that is not listed above. Then the maps defined above on the graded vector space V form an A_∞ algebra.

Proof (sketch)

We may first try to verify the identity

$$\sum_{\lambda=0}^{n-1}\sum_{k=1}^{n-\lambda}\alpha m_{n-k+1}(x_1\otimes\cdots\otimes x_\lambda\otimes m_k(x_{\lambda+1}\otimes\cdots\otimes x_{\lambda+k})\otimes x_{\lambda+k+1}\otimes\cdots\otimes x_n)$$

where $\alpha=(-1)^{k+\lambda+k\lambda+kn+k(|x_1|+\cdots+|x_\lambda|)}$, for all $n\in\mathbb{N}$, $x_i\in V$.

Problem: This is no easy task due to the varying signs, s_n, accompanying the m_n maps!

An alternative approach

J. Stasheff: An A_∞ structure on a graded vector space V is equivalent to the existence of a degree 1 coderivation $D : T^* \downarrow V \to T^* \downarrow V$ with the property $D^2 = 0$, where $\downarrow V$ denotes the desuspension of V . Such a coderivation is constructed by defining

$$D := \sum_{k=1}^{\infty} m'_k$$

where $m'_k :\downarrow V^{\otimes k} \to \downarrow V$ is given by $m'_k := (-1)^{\frac{k(k-1)}{2}} \downarrow \circ m_k \circ \uparrow^{\otimes k}$

and is extended as a coderivation.

An alternative approach

- Signs become much more manageable:

$$
\begin{aligned}
m'_1 &= \downarrow m_1 \\
\textit{For } n \geq 2: \quad m'_n(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2)-k}) &= \downarrow v_1, \ 0 \leq k \leq n-2 \\
m'_n(\downarrow v_1 \otimes \downarrow w^{\otimes (n-2)} \otimes \downarrow v_2) &= \downarrow v_1 \\
m'_n(\downarrow v_1 \otimes \downarrow w^{\otimes (n-1)}) &= \downarrow w
\end{aligned}
$$

- Proof strategy: Show $D^2 = 0$ by induction on the number of inputs for D (i.e. $D^2(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) = 0 \ \forall \ n$).

Further investigation

Now that we have a concrete example of an A_∞ algebra structure, many paths are open to us

- Daily/Lada's L_∞ example has a slick interpretation related to gauge theory, and is also an example of an OCHA (Open-Closed Homotopy Algebra). Can we apply our A_∞ algebra example in a similar fashion?
- Using commutators, what new L_∞ algebra structure will our example induce on the same graded vector space?