A Finite Dimensional $A_\infty$ Algebra Example

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$A_\infty$ algebras

**Definition**
Let $V$ be a graded vector space. An $A_\infty$ algebra on $V$ is a collection of linear maps $m_k : V \otimes^k \rightarrow V$ of degree $2 - k$ that satisfy the identity

$$\sum_{\lambda=0}^{n-1} \sum_{k=1}^{n-\lambda} \alpha m_{n-k+1}(x_1 \otimes \cdots \otimes x_\lambda \otimes m_k(x_{\lambda+1} \otimes \cdots \otimes x_{\lambda+k}) \otimes x_{\lambda+k+1} \otimes \cdots \otimes x_n) = 0$$

where $\alpha = (-1)^{k+\lambda+k\lambda+kn+k(|x_1|+\cdots+|x_\lambda|)}$, for all $n \geq 1$.

- For $n = 3$ in a 1-term graded vector space, this yields
  $$m_3(m_1(x_1), x_2, x_3) + (-1)^{|x_1|}m_3(x_1, m_1(x_2), x_3) + (-1)^{|x_1||x_2|}m_3(x_1, x_2, m_1(x_3))$$
  $$+ m_2(m_2(x_1, x_2), x_3) - m_2(x_1, m_2(x_2, x_3)) + m_1(m_3(x_1, x_2, x_3)) = 0$$

- With $m_2$ acting as multiplication, this behaves like an associative algebra.
**$L_\infty$ algebras**

**Definition**
Let $V$ be a graded vector space. An $L_\infty$ algebra on $V$ is a collection of skew symmetric linear maps $l_n : V^\otimes n \to V$ of degree $2 - n$ that satisfy the identity

$$
\sum_{i+j=n+1} \sum_{\sigma} e(\sigma)(-1)^{\sigma} (-1)^{i(j-1)} l_j(l_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}) = 0
$$

where $(-1)^{\sigma}$ is the sign of the permutation, $e(\sigma)$ is the sign that arises from the degrees of the permuted elements, and $\sigma$ is taken over all $(i, n-i)$ unshuffles.

- For $n = 3$ in a 1-term graded vector space, this yields

$$
l_3(l_1(x_1), x_2, x_3) - (-1)^{|x_1||x_2|} l_3(l_1(x_2), x_1, x_3) + (-1)^{|x_3|(|x_1|+|x_2|)} l_3(l_1(x_3), x_1, x_2)
$$

$$
+ l_2(l_2(x_1, x_2), x_3) + (-1)^{|x_2||x_3|} l_2(l_2(x_1, x_3), x_2))
$$

$$
+ (-1)^{|x_1|(|x_2|+|x_3|)} l_2(l_2(x_2, x_3), x_1) + l_1(l_3(x_1, x_2, x_3)) = 0
$$

- With $l_2$ representing a Lie bracket, this behaves just like a Lie algebra.
Slick relationship

Recall: Given an associative algebra $A$ with multiplication $\cdot$, $A$ induces a Lie algebra through the commutator bracket: $[x, y] = x \cdot y - y \cdot x$

Similarly, an $A_\infty$ algebra will induce an $L_\infty$ algebra through its commutators:

- $l_1 := m_1$
- $l_2(x, y) := m_2(x, y) - (-1)^{|x||y|} m_2(y, x)$
- $l_3(x, y, z) := m_3(x, y, z) - (-1)^{|x||y|} m_3(y, x, z) - (-1)^{|y||z|} m_3(x, z, y) - (-1)^{|x||y|+|x||z|+|y||z|} m_3(z, y, x) + (-1)^{|z||x||y|} m_3(z, x, y) + (-1)^{|x||y||z|} m_3(y, z, x) = 0$
- etc..
Concrete examples are far from trivial

Problem: Due to the graded setting and infinite amount of maps, even construction of a simple ‘interesting’ example of an $A_\infty$ or $L_\infty$ algebra is difficult!

Marilyn Daily and Tom Lada constructed a finite dimensional $L_\infty$ algebra example (2004).
A finite dimensional \( L_\infty \) algebra example

Example (Daily, Lada)
Consider the graded vector space \( V = \bigoplus V_n \) where \( V_0 \) is a 2-dimensional space with basis \( \langle v_1, v_2 \rangle \) and \( V_1 \) is a 1-dimensional space with basis \( \langle w \rangle \). Let \( V_n = 0 \) for \( n \neq 0, 1 \). We define an \( L_\infty \) structure, \( l_n : V^{\otimes n} \to V \), on \( V \) via the following maps:

\[
\begin{align*}
l_1(v_1) &= l_1(v_2) = w \\
l_2(v_1 \otimes v_2) &= v_1, \quad l_2(v_1 \otimes w) = w \\
l_n(v_2 \otimes w^{\otimes n-1}) &= C_n w \quad \text{for all } n \geq 3
\end{align*}
\]

where \( C_n = (-1)^{\frac{(n-2)(n-3)}{2}} (n-3)! \). We extend these maps to be skew symmetric and define \( l_n \) to be 0 when evaluated on any element of \( V^{\otimes n} \) that is not listed above.
What about a similar $A_{\infty}$ algebra example?

Motivating question: Can we build an $A_{\infty}$ algebra structure with the following properties?

1. Defined over the same graded vector space
2. Commutator induces Daily/Lada’s $L_{\infty}$ structure

Property 2 adds great difficulty in generating such a structure. For ease of computation, we only consider property 1.
A finite dimensional example

Theorem
Let $V$ denote the graded vector space given by $V = \bigoplus V_n$ where $V_0 = \langle v_1, v_2 \rangle$, $V_1 = \langle w \rangle$, and $V_n = 0$ for $n \neq 0, 1$. Define a structure on $V$ by the following linear maps $m_n : V^\otimes n \to V$:

$m_1(v_1) = m_1(v_2) = w$

For $n \geq 2$:

$m_n(v_1 \otimes w^\otimes k \otimes v_1 \otimes w^\otimes (n-2)-k) = (-1)^k s_n v_1$, $0 \leq k \leq n - 2$

$m_n(v_1 \otimes w^\otimes (n-2) \otimes v_2) = s_{n+1} v_1$

$m_n(v_1 \otimes w^\otimes (n-1)) = s_{n+1} w$

Where $s_n = (-1)^{\frac{(n+1)(n+2)}{2}}$, and $m_n = 0$ when evaluated on any element of $V^\otimes n$ that is not listed above. Then the maps defined above on the graded vector space $V$ form an $A_\infty$ algebra.
Proof (sketch)

We may first try to verify the identity

$$\sum_{\lambda=0}^{n-1} \sum_{k=1}^{n-\lambda} \alpha m_{n-k+1}(x_1 \otimes \cdots \otimes x_\lambda \otimes m_k(x_{\lambda+1} \otimes \cdots \otimes x_{\lambda+k}) \otimes x_{\lambda+k+1} \otimes \cdots \otimes x_n)$$

where $\alpha = (-1)^{k+\lambda+k\lambda+kn+k(|x_1|+\cdots+|x_\lambda|)}$, for all $n \in \mathbb{N}$, $x_i \in V$.

Problem: This is no easy task due to the varying signs, $s_n$, accompanying the $m_n$ maps!
An alternative approach

J. Stasheff: An $A_\infty$ structure on a graded vector space $V$ is equivalent to the existence of a degree 1 coderivation $D : T^* \downarrow V \to T^* \downarrow V$ with the property $D^2 = 0$, where $\downarrow V$ denotes the desuspension of $V$. Such a coderivation is constructed by defining

$$D := \sum_{k=1}^{\infty} m'_k$$

where $m'_k : \downarrow V^k \to \downarrow V$ is given by $m'_k := (-1)^{\frac{k(k-1)}{2}} \downarrow \circ m_k \circ \uparrow^k$ and is extended as a coderivation.
An alternative approach

- Signs become much more manageable:

  \[ m'_1 = \downarrow m_1 \]

  For \( n \geq 2 \):
  \[
  m'_n(\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2) - k}) = \downarrow v_1, \quad 0 \leq k \leq n - 2
  \]
  \[
  m'_n(\downarrow v_1 \otimes \downarrow w^{\otimes (n-2)} \otimes \downarrow v_2) = \downarrow v_1
  \]
  \[
  m'_n(\downarrow v_1 \otimes \downarrow w^{\otimes (n-1)}) = \downarrow w
  \]

- Proof strategy: Show \( D^2 = 0 \) by induction on the number of inputs for \( D \) (i.e. \( D^2(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) = 0 \ \forall \ n \)).
Further investigation

Now that we have a concrete example of an $A_\infty$ algebra structure, many paths are open to us

- Daily/Lada’s $L_\infty$ example has a slick interpretation related to gauge theory, and is also an example of an OCHA (Open-Closed Homotopy Algebra). Can we apply our $A_\infty$ algebra example in a similar fashion?
- Using commutators, what new $L_\infty$ algebra structure will our example induce on the same graded vector space?