

Quantum Master  
Equation (QME) of  
Open-Closed (OC)  
String Theory

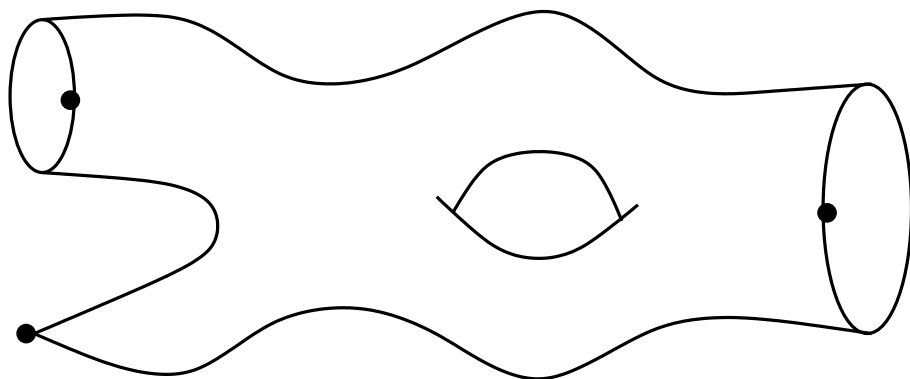
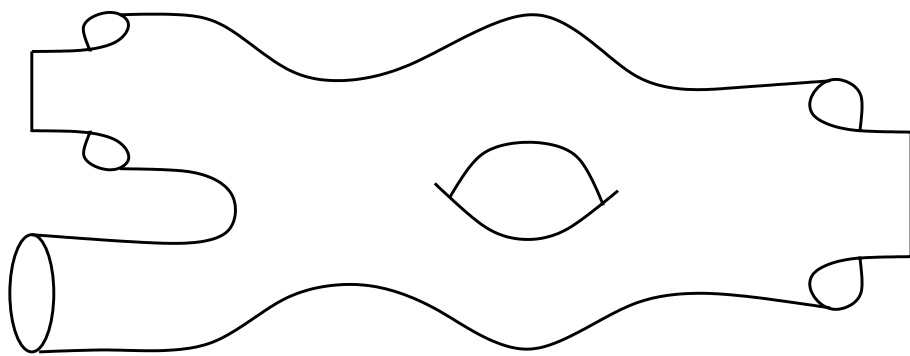
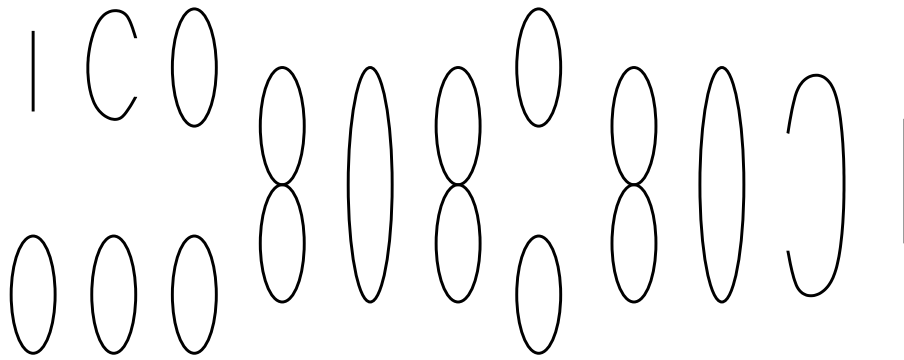
joint with E. Harrelson and A. Voronov

## REFERENCES

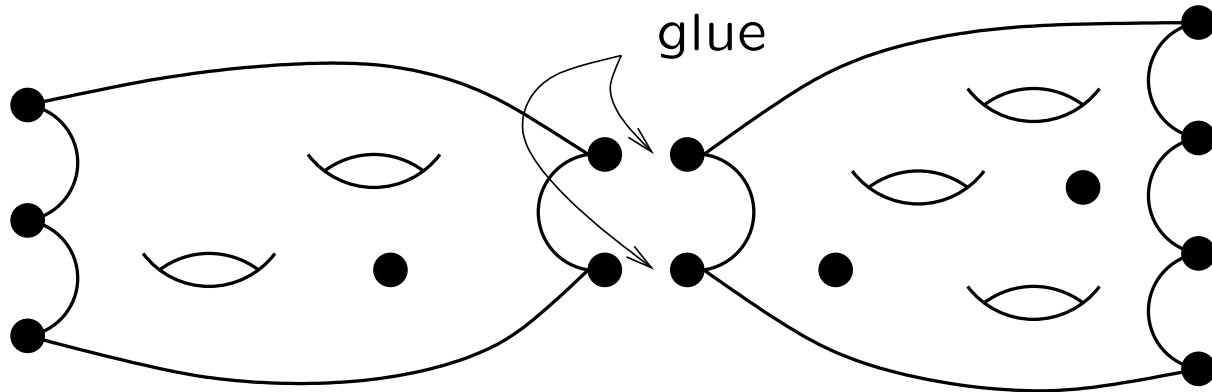
Kimura, Stasheff, and Voronov; *Homology of moduli of curves and commutative homotopy algebras*, 1996

Liu; *Moduli of J-Holomorphic Curves with Lagrangian Boundary Conditions and Open Gromov-Witten Invariants for an  $S^1$ -Equivariant Pair*, 2002

# OVERVIEW



Need to consider Moduli Spaces of R.S.'s

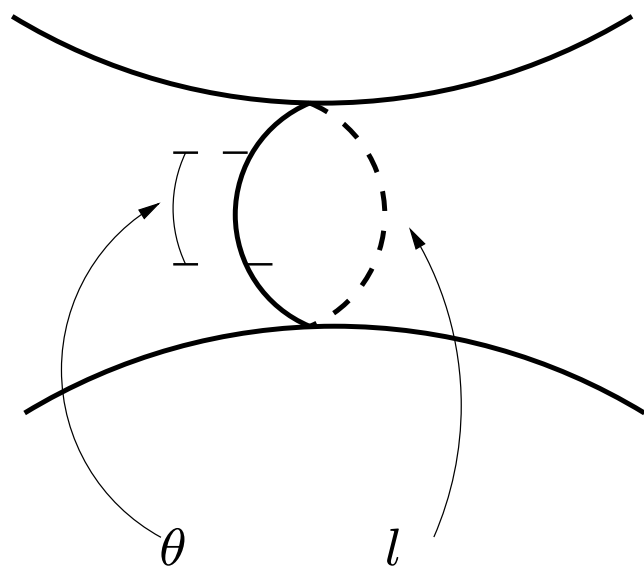


### Composition of Surfaces

We need to consider compactification of Moduli Spaces with simple degenerations.

Remark.- Degeneration of closed geodesics kills two parameters:

- 1)  $l$ =length
- 2)  $\theta$ =angle



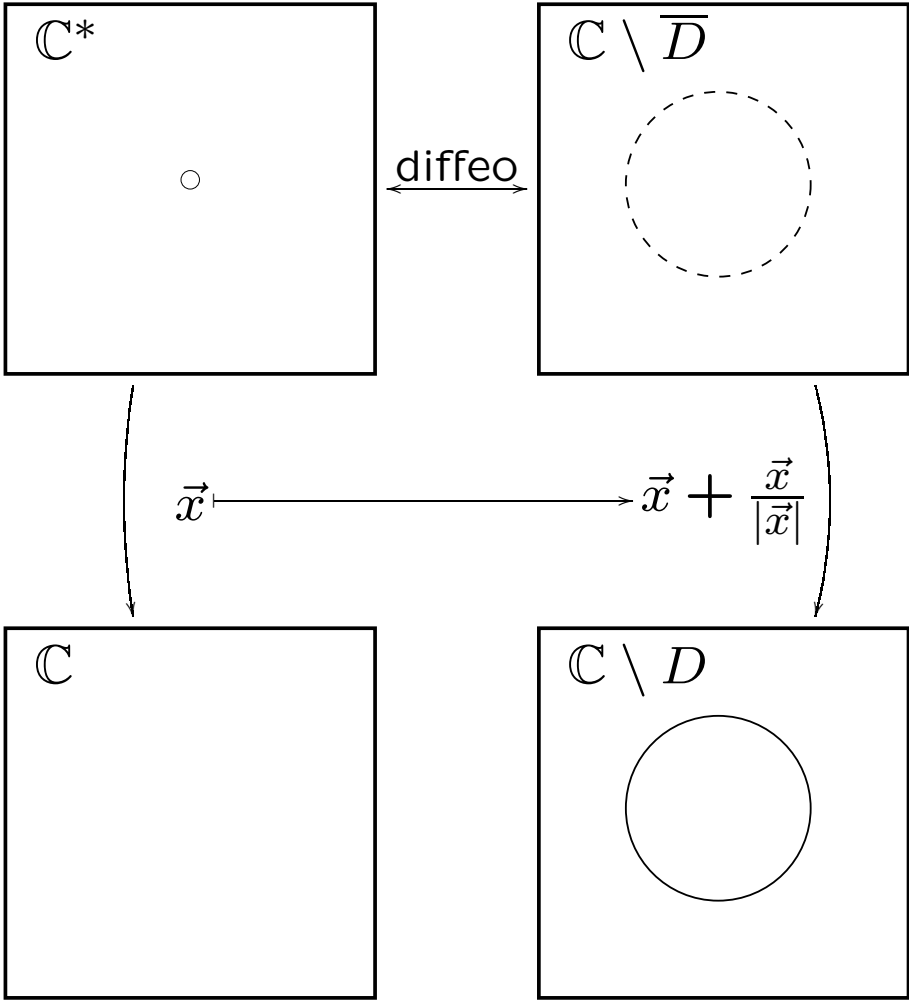
QME:

$$dS + \Delta S + \frac{1}{2}\{S, S\} = 0$$

A single equation which describes all the strata of a certain moduli space and how they fit together using an algebraic structure.

Remark.- The differential  $d$  is of codim 1 and the closed degenerations give codim 2 spaces.

Two ways to compactify

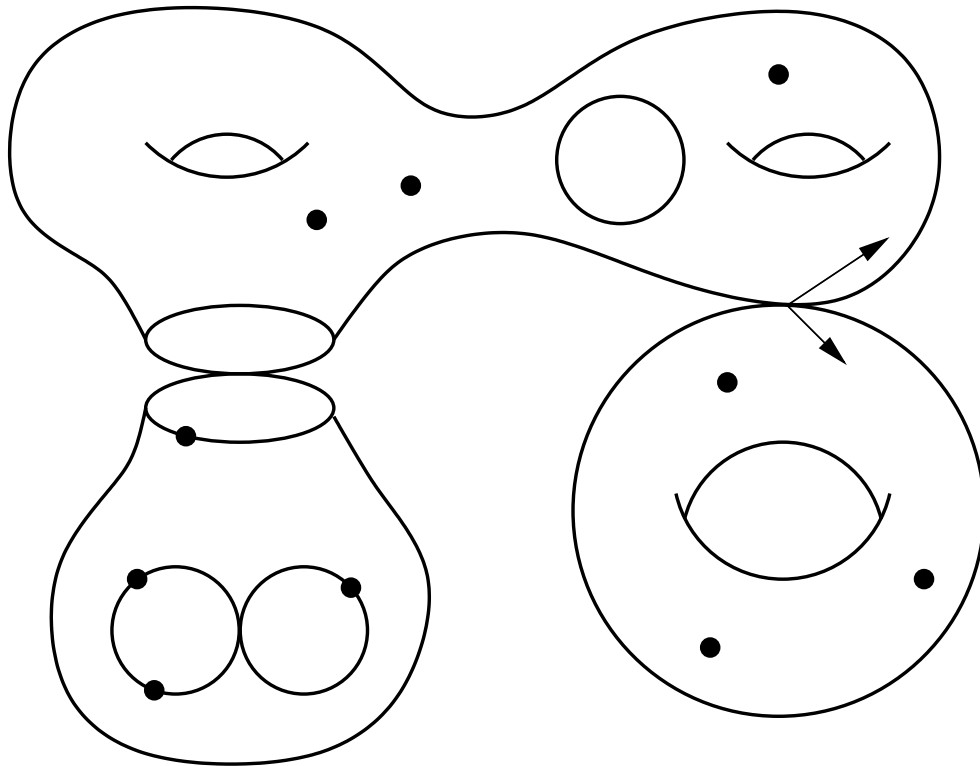


D-M compactification (codim-2)

Directional blow-up (codim-1)

## OC MODULI SPACES

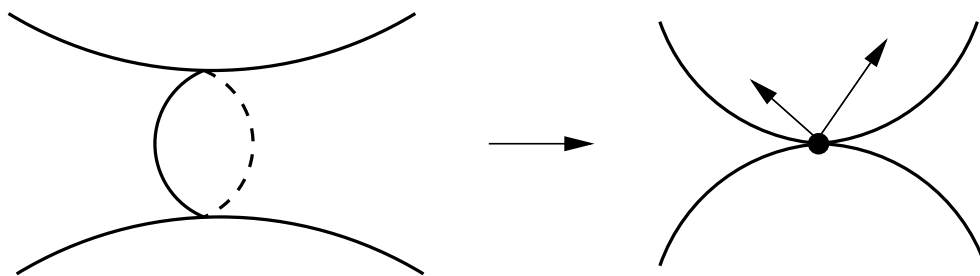
$\mathcal{M}_{g,b}^{n,\vec{m}}$  = Moduli space of connected stable bordered R.S.'s of genus  $g$  with  $b$  boundary components,  $n$  ordered interior punctures and  $\vec{m} = (m_1, \dots, m_b)$  boundary punctures ordered within each boundary component in a way compatible with its orientation.



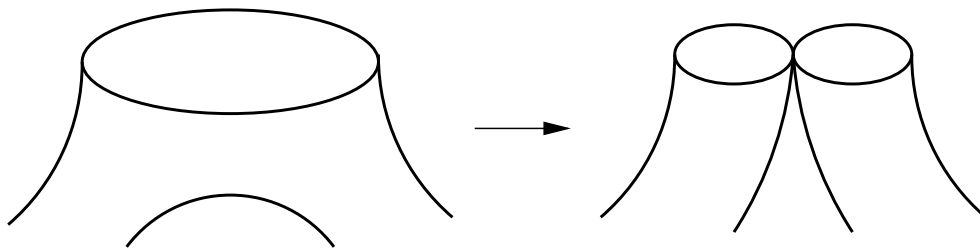
Here  $g = 3$ ,  $b = 4$ ,  $n = 5$ ,  $\vec{m} = (0, 3, 1, 0)$

Degenerations: 3 types allowed

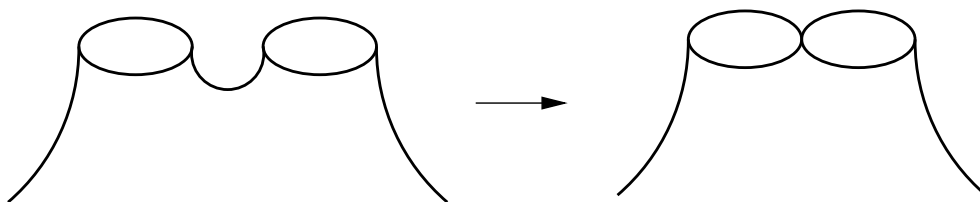
X: Here the pair of arrows represent extra data at double point, a real ray in  $T_p^1 \otimes_{\mathbb{C}} T_p^2$ .



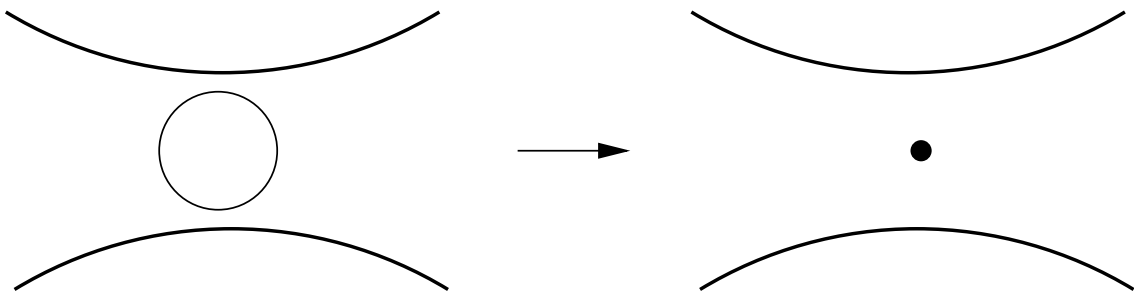
H:



or



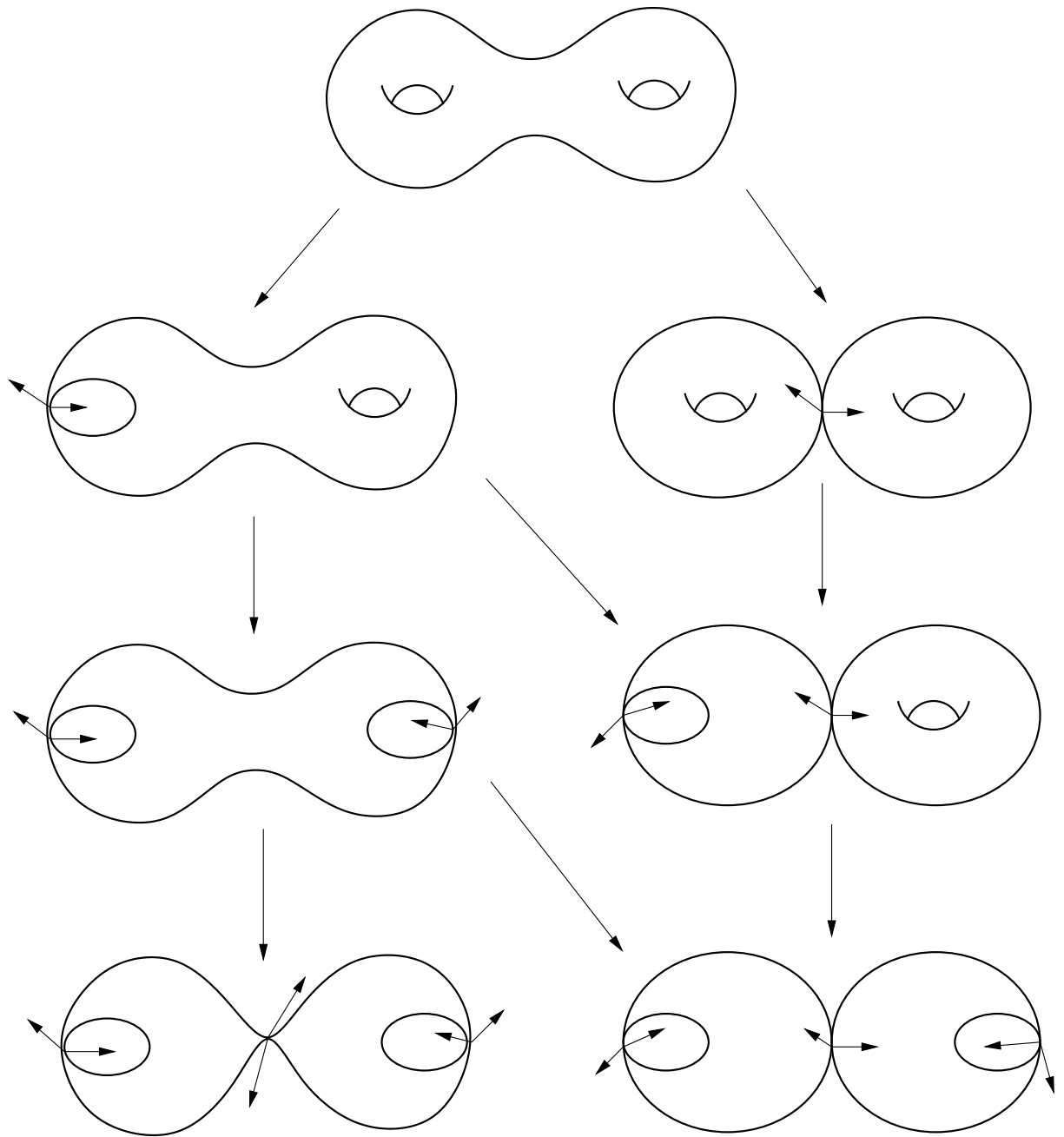
E: boundary component collapsed to a point  
(closed open string interaction)



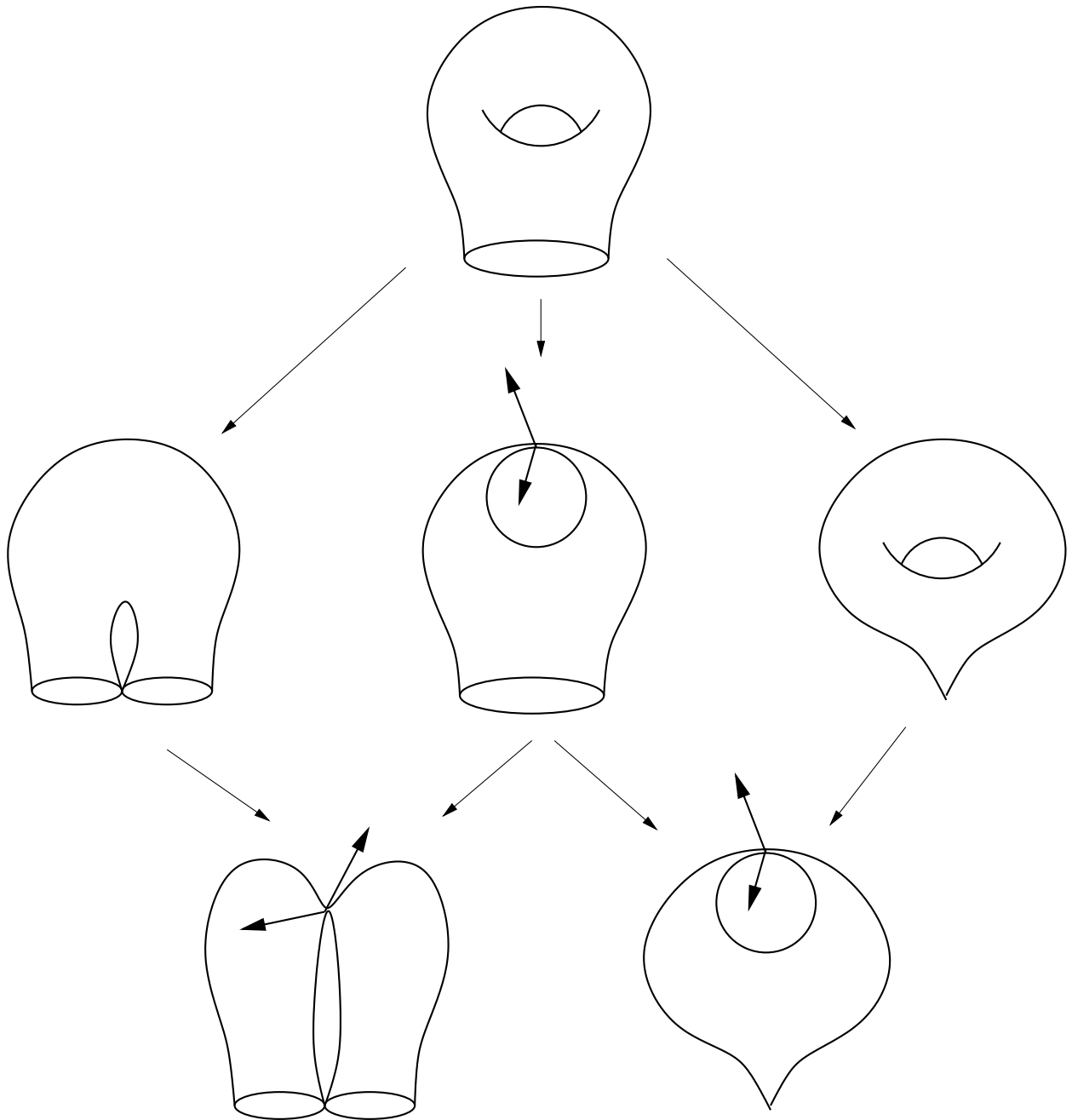
Theorem.- (corollary of Liu's thm)  $\underline{\mathcal{M}}_{g,b}^{n,\vec{m}}$  is an oriented compact orbifold with corners, moreover

$$\dim \underline{\mathcal{M}}_{g,b}^{n,\vec{m}} = 6g - 6 + 3b + 2n + \sum_{i=1}^b m_i$$

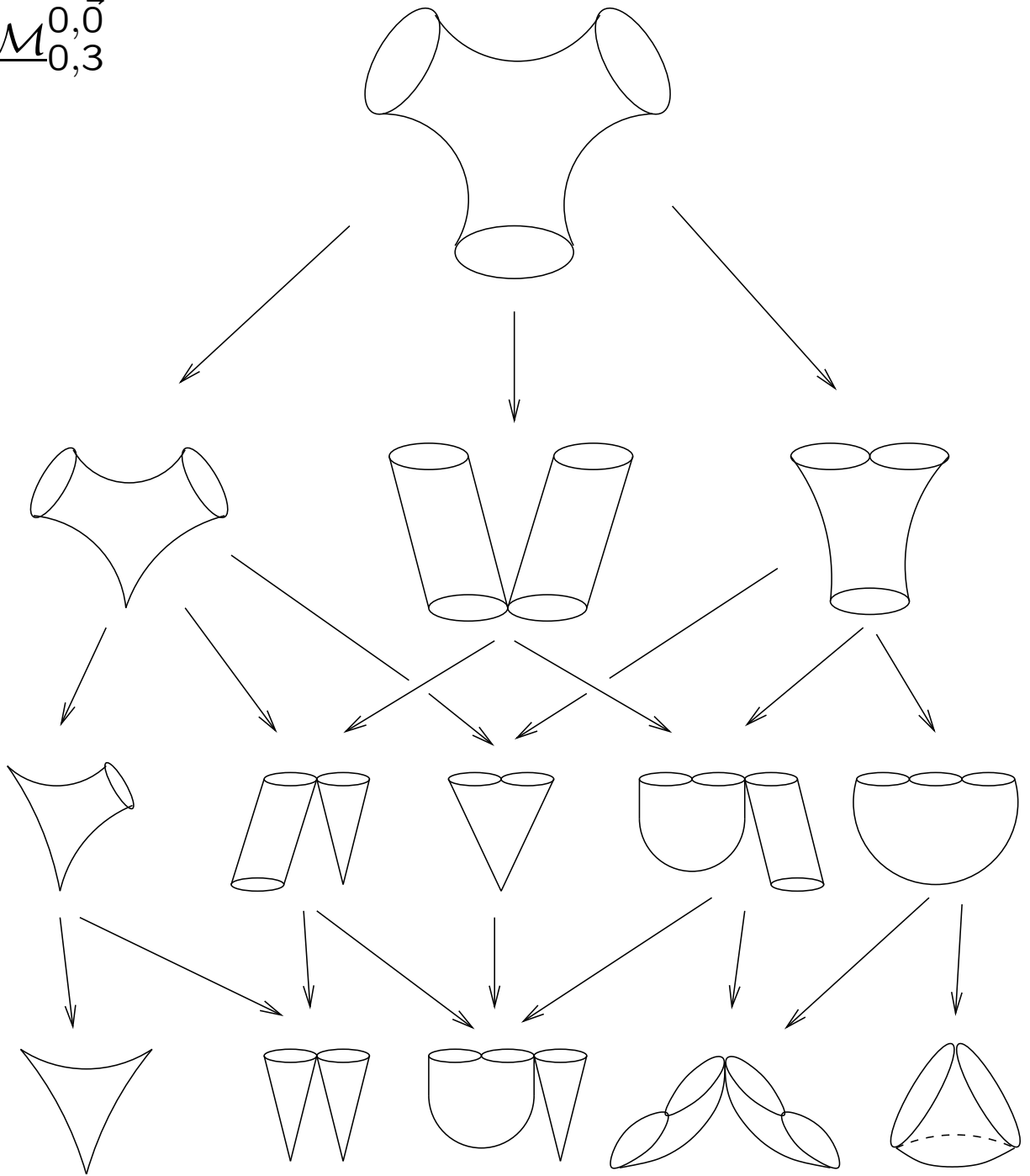
$\mathcal{M}_{2,0}^{0,0}$



$\mathcal{M}_{1,1}^{0,\vec{0}}$



$\mathcal{M}_{0,3}^{0,\vec{0}}$



Actually  $\mathcal{M}_{0,3}^{0,\vec{0}} \cong K_5$

## A DG VECTOR SPACE

$$\underline{\mathcal{M}}_{g,b}^{n,m} / \mathfrak{S}$$

||

$$\left( \coprod_{\substack{\vec{m} \\ \sum m_i = m}} \left( \underline{\mathcal{M}}_{g,b}^{n,\vec{m}} / \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_b} \right) \right) / \mathfrak{S}_b \times \mathfrak{S}_n$$

- $\underline{\mathcal{M}}_b^{n,m} / \mathfrak{S} =$  Moduli space of possibly disconnected surfaces of several genus.
- $C_{\bullet}^{geom}(-, \mathcal{F}) =$  Geometric chains with coefficients in a locally constant sheaf of  $\mathbb{Q}$ -vector spaces

Definition.- A geometric chain on  $X$  is a l. c. over  $\mathbb{Q}$  of continuous maps  $f : P \rightarrow X$  where  $P$  is a compact connected oriented smooth orbifold with corners, modulo isomorphisms between the source orbifolds  $P$ .  $C_{\bullet}^{geom}(X)$  is the DG  $\mathbb{Q}$ -vector space graded by  $\dim P$  and  $d(f, P) = (f|_{\partial P}, \partial P)$ . This way we get:

- Compact orbifolds inherit natural fundamental chains (the class of the identity map).
- Given  $A \subset X$  restricting chains to satisfy  $f(\partial P) \subset A$  produces relative homology classes  $f_*[P] \in H_{\bullet}(X, A; \mathbb{Q})$  where  $[P]$  is the orbifold fundamental chain.
- Given a locally constant sheaf  $\mathcal{F}$  of  $\mathbb{Q}$ -vector spaces on  $X$  we can define geometric chains with coeff. in  $\mathcal{F}$  as a l.c.  $c = \sum c_i(P_i, f_i)$  where  $c_i \in \Gamma(P_i; f_i^* \mathcal{F})$

$$V = S \left( \bigoplus_{g,b,m,n} C_{\bullet}^{geom} (\underline{\mathcal{M}}_{g,b}^{n,m} / \mathfrak{S}; \mathbb{Q}^{\epsilon}) \right)$$

$$\subset_{\text{dense}} \bigoplus_{b,m,n} C_{\bullet}^{geom} (\underline{\mathcal{M}}_b^{n,m} / \mathfrak{S}; \mathbb{Q}^{\epsilon})$$

where  $\mathbb{Q}^{\epsilon}$  is a local coefficient system with

- $\zeta \in \mathbb{Z}_m$  acting by  $(-1)^{m-1}$
- $\tau_{ij} \in \mathfrak{S}_b$  acting by  $(-1)^{(m_i+1)(m_j+1)}$
- $\mathfrak{S}_n$  acting trivially

Proposition.-  $\mathbb{Q}^{\epsilon}$  is the orientation sheaf for  $\underline{\mathcal{M}}_{g,b}^{n,m}$  and thus we get the fundamental chain

$$[\underline{\mathcal{M}}_{g,b}^{n,m} / \mathfrak{S}] \in C_{6g-6+2n+3b+m}^{geom} (\underline{\mathcal{M}}_{g,b}^{n,m} / \mathfrak{S}; \mathbb{Q}^{\epsilon})$$

Grading: by codimension relative to the corresponding moduli space  $\underline{\mathcal{M}}_{g,b}^{n,m}$ .

Differential: induced from the differential on geometric chains, thus we have  $d : V^k \rightarrow V^{k+1}$  and  $d^2 = 0$ .

## BV STRUCTURE IN $V$

Definition.- A DG-BV algebra (here) is a DG commutative algebra ( $\deg d=1, d(1) = 0$ ) with a second order differential  $\Delta$  ( $\Delta^2 = 0, \Delta(1) = 0, \deg \Delta = 1, [d, \Delta] = 0$ )

Remark.- We get an anti-bracket by letting

$$\{a, b\} = \pm \Delta(a \cdot b) \mp \Delta(a) \cdot b \mp a \cdot \Delta(b)$$

So in  $V$ :

- Dot product induced by disjoint union of surfaces.

- $\Delta = \Delta_c + \Delta_o + \Delta_{co}$  where

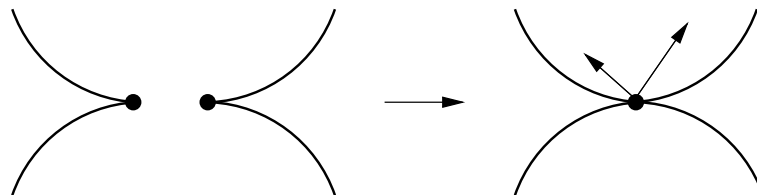
$$- \Delta_c^2 = \Delta_o^2 = \Delta_{co}^2 = 0$$

$$[\Delta_c, \Delta_o] = [\Delta_c, \Delta_{co}] = [\Delta_o, \Delta_{co}] = 0$$

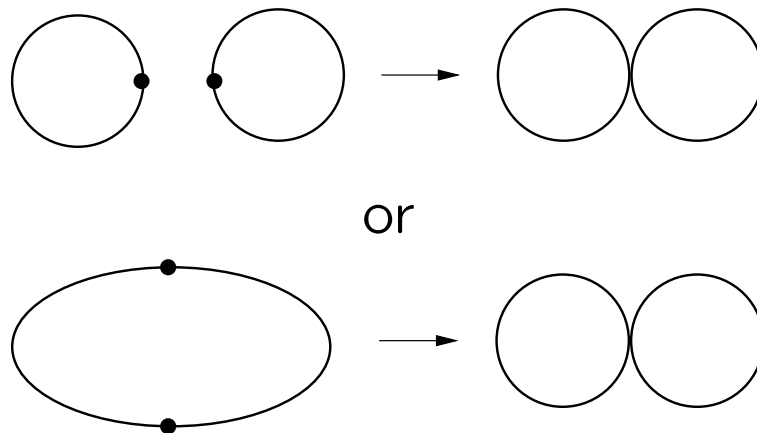
$\Delta_c, \Delta_o$  are second order D.O.'s

$\Delta_{co}$  is a first order derivation

- $\Delta_c$  induced by twist-attaching a pair of interior punctures and adding up over all pairs



- $\Delta_o$  induced by attaching pairs of boundary components



- $\Delta_{co}$  induced by declaring an interior punc. to be a deg. boundary component

## QME

$$dS + \Delta S + \frac{1}{2}\{S, S\} = 0$$

In the BV algebra  $V[[\lambda]]$  let

$$S = \sum_{\substack{g,b \\ n,m}} [\underline{\mathcal{M}}_{g,b}^{n,m} / \mathfrak{G}] \lambda^{-2\chi}$$

where  $\chi = 2 - 2g - n - b - \frac{m}{2}$ .

Theorem.-  $S$  is a solution of QME

Rough Idea: The QME states that the boundary of the fundamental chain  $[\underline{\mathcal{M}}_{g,b}^{n,m} / \mathfrak{G}]$  is comprised of lower-dimensional fundamental chains obtained by attaching punctures to each other on a single surface ( $\Delta_c S$  and  $\Delta_o S$ ) or on different surfaces ( $\{S, S\}$ ) or degenerating interior punctures into degenerated boundary components ( $\Delta_{co} S$ ).

Definition.- A “symmetric” TCFT is a BV-algebra morphism from our BV-algebra  $V[[\lambda]]$  to another one.

Idea:

preserves  $\bullet \Leftrightarrow$  behaves well under  $\coprod$  of surfaces

preserves  $\Delta \Leftrightarrow$  behaves well under gluing (factorization axiom) or closed-open string interaction

Remark.- Any symmetric TCFT comes with a solution to QME, namely, the image of  $S$ .

Example.- The BV-algebra for  $\sigma$ -model must be  $S^\bullet(\oplus_{i,j \in D} H_{ij} \oplus H_c)$

Our  $S$  may be viewed as a  $\sigma$ -model with target space  $M = \{\text{pt.}\}$

On the other hand, it is the universal solution to QME.