Field Equations from
Homotopy Algebras of CFT

Anton M. Zeitlin

AMS Sectional Meeting (Raleigh, NC)
Special Session on Homotopical Algebra
with
Applications to Mathematical Physics
2009
• Motivation

• Reminder of Lian-Zuckerman (LZ) homotopy algebras

• Relation of LZ constructions to perturbed CFTs and $\sigma$-models

• Examples:
  – Einstein equations
  – Yang-Mills equations
  – Kodaira-Spencer equations via chiral de Rham complex

• Conclusions
Motivation

String theory:
2d Conformal Field Theory $\rightarrow$
D-dimensional Quantum Field Theory

Linear classical field equations (Maxwell, Linearized Einstein) and their symmetries:

$$Q \Phi = 0, \quad \Phi \rightarrow \Phi + Q \lambda$$

$Q$ is a semi-infinite cohomology operator for the Virasoro algebra.

What about nonlinear equations?

String Field Theory:

$$Q \Phi + \sum_n \mu_n(\Phi, \ldots, \Phi) = 0$$

$Q, \mu_n : A_\infty / L_\infty$

The description of $\mu_n$ is too complicated. Something more explicit?
Natural mathematical tool to study 2d CFT: VOA (vertex operator algebra)

B.H. Lian, G.J. Zuckerman
"New Perspectives on the BRST-algebraic Structure of String Theory"

LZ homotopy algebra (reminder)

Let $V$ be the VOA. $T(z)$ is a Virasoro element

Semi-infinite complex : $C^* = V \otimes \Lambda$

$\Lambda$ - VOA of “conformal ghosts”:

$$b(z)c(w) \sim \frac{1}{z-w}$$

$$Q = \oint dz \ c(z)T(z)+ :c\partial c:b \ (z)$$

semi-infinite cohomology (BRST) operator.

Let $a(z)$ be a vertex operator for $a \in V \otimes \Lambda$

Operations:

$$\mu(a_1, a_2) \equiv P_0 a_1(\varepsilon) a_2$$

$$\{a_1, a_2\} \equiv (-1)^{|a_1|} \oint dz (b_{-1} a_1)(z) a_2$$

where $P_0$ is the projection on $\varepsilon$ - independent component.
Proposition 1

(i) \[ \mu(a_1, a_2) - (-1)^{|a_1||a_2|} \mu(a_2, a_1) = \]
\[ Qm(a_1, a_2) + m(Qa_1, a_2) + (-1)^{|a_1|}m(a_1, Qa_2) \]

(ii) \[ \mu(\mu(a_1, a_2), a_3) - \mu(a_1, \mu(a_2, a_3)) = \]
\[ Qn(a_1, a_2, a_3) + n(Qa_1, a_2, a_3) + (-1)^{|a_1|}n(a_1, Qa_2, a_3) \]
\[ + (-1)^{|a_1|+|a_2|}n(a_1, a_2, Qa_3), \]

Lemma

\[ (-1)^{|a_2|}\{a_1, a_2\} = b_0\mu(a_1, a_2) - \mu(b_0a_1, a_2) - (-1)^{|a_2|}\mu(a_1, b_0a_2) \]

Proposition 2

(i) \[ \{a_1, a_2\} + (-1)^{|a_1|-1)(|a_2|-1}\{a_2, a_1\} = \]
\[ (-1)^{|a_1|-1}(Qm'(a_1, a_2) + m'(Qa_1, a_2) \]
\[ - (-1)^{|a_2|}m'(a_1, Qa_2)) \]

(ii) \[ \{\{a_1, a_2\}, a_3\} - \{a_1, \{a_2, a_3\}\} \]
\[ + (-1)^{|a_1|-1)(|a_2|-1}\{a_2, \{a_1, a_3\}\} = 0 \]

(iii) \[ \{a_1, \mu(a_2, a_3)\} = \mu(\{a_1, a_2\}, a_3) \]
\[ + (-1)^{|a_1|-1)|a_2|\mu(a_2, \{a_1, a_3\}) \]

(iv) \[ \{\mu(a_1, a_2), a_3\} - \mu(a_1, \{a_2, a_3\}) \]
\[ - (-1)^{|a_3|-1)|a_2|\mu(\{a_1, a_3\}, a_2) = \]
\[ (-1)^{|a_1|+|a_2|-1}(Qn''(a_1, a_2, a_3) - n''(Qa_1, a_2, a_3) - \]
\[ (-1)^{|a_2|}n''(a_1, Qa_2, a_3) - (-1)^{|a_1|+|a_2|}n''(a_1, a_2, Qa_3)) \]
Therefore $V \otimes \Lambda$ carry a structure of homotopy Gerstenhaber algebra.

One can generalize further:

$$C^* = C^* \otimes \bar{C}^* \quad \mu^\text{ext}(a_1, a_2) = P_0 a_1(\epsilon)a_2, \quad \epsilon \notin \mathbb{R}$$

$$\{a_1, a_2\}^\text{ext} = P_0 \int_{C_{\varepsilon,0}} a_1^{(1)} a_2,$$

where $a^{(1)} = d_z (b_{-1} a)(z) + d \bar{z} (\bar{b}_{-1} a)(\bar{z}).$

$\mu^\text{ext}, \{\cdot, \cdot\}^\text{ext}$ also satisfy the relations of homotopy Gerstenhaber algebra w.r.t. operator $Q = Q + \bar{Q}$

Conjecture (L-Z) Operations $\mu, \{\cdot, \cdot\}$ and $\mu^\text{ext}, \{\cdot, \cdot\}^\text{ext}$ can be extended to the structure of $G_\infty$ algebra.

Attempts to prove:

Kimura, Voronov, Zuckerman
“Homotopy Gerstenhaber algebra and topological field theory”
q-alg/9602009

Galvez, Gorbounov, Tonks
“Homotopy Gerstenhaber Structures and Vertex Algebras”
math/0611231
What is the physical meaning of the associated Maurer-Cartan equations?

\[ Q\Phi + \mu(\Phi, \Phi) + \sum_{n=3}^{\infty} \mu_n(\Phi, \ldots, \Phi) = 0 \]

\[ Q\Psi + \frac{1}{2}\{\Psi, \Psi\} + \sum_{n=3}^{\infty} \frac{1}{n!}\{\Psi, \ldots, \Psi\}_n = 0 \]

We will show that

a) they lead to nonlinear field equations and their symmetries

b) can give rise to an “algebraic” definition of \( \beta \)-function for the perturbed CFT (\( \sigma \)-models in particular)
Perturbed CFTs

VOA $\rightarrow$ CFT with some action $S_0$.

Perturbations: $S_0 \rightarrow S = S_0 + V$

\[ V = \int_{\Sigma} \Phi^{(2)}, \quad \Phi^{(2)} = dz \wedge d\bar{z} A(z), \quad A \in C^* \otimes \bar{C}^* \]

In general, the perturbed theory is not a CFT. Renormalization theory gives the condition for the theory to be conformal:

\[ \beta(\Phi) = 0 \]

Expand: $\Phi^{(2)} = t\Phi_1^{(2)} + t^2\Phi_2^{(2)} + \ldots$

\[ \beta_1(V) = 0 \iff Q\Phi_1^{(0)} = 0 \]

\[ \beta_2(V) = 0 \iff Q\Phi_2^{(0)} + \frac{1}{4\pi i} P_0 \oint_{C_{\varepsilon,0}} \Phi_1^{(1)} \Phi_1^{(0)} = 0 \]

where $\Phi_i^{(2)} = dz \wedge d\bar{z}[b_{-1}, [\bar{b}_{-1}, \Phi_i^{(0)}]]$

A. Sen (89,90);

A.M.Z.

“BRST, Generalized Maurer-Cartan Equations and CFT”,

“Formal Maurer-Cartan Structures: From CFT to Classical Field Equations”,
JHEP0709:098(2007)
Therefore one can replace $\beta(\Phi)$ by

$$\hat{\beta}(\Phi) = Q\Phi + \frac{1}{2}\{\Phi, \Phi\} + \sum_{n=3}^{\infty} \frac{1}{n!}\{\Phi, \ldots, \Phi\}_n$$

where $\Phi^{(2)} = dz \wedge \bar{d}z[b_{-1}, [\bar{b}_{-1}, \Phi]]$

One can think of $\hat{\beta}(\Phi)$ as an “algebraic” definition of $\beta$-function.

The same applies to $\beta$-functions for the theories with boundary perturbations:

$$S_0 \longrightarrow S = S_0 + \int_{\partial \Sigma} \Phi^{(1)}$$

In this case ”algebraic” $\beta$-function is:

$$\beta(\Phi) = Q\Phi + \mu(\Phi, \Phi) + \sum_{n=3}^{\infty} \mu_n(\Phi, \ldots, \Phi)$$

where $\Phi^{(1)} = dz[b_{-1}, \Phi] + \bar{d}z[\bar{b}_{-1}, \Phi]$
\( \sigma \)-models

\[
S = \frac{1}{2\pi h} \int_{\Sigma} d^2z (G_{\mu\nu}(X) + B_{\mu\nu}(X)) \partial X^\mu \bar{\partial} X^\nu \\
+ \frac{1}{2\pi} \int R^{(2)} \phi(X) + \int_{\partial \Sigma} A_\mu(X) dX^\mu
\]

\[
\beta(G, B, \phi, A) = \sum_{n=1}^{\infty} h^n \beta_n(G, B, \phi, A)
\]

\( \beta_1 = 0 \iff \text{Einstein-Yang-Mills equations} \)

\( \beta_n = 0 \ (n > 1) \) gives equations with higher derivatives

Complications:

i) \( S = S_0 + V \) destroys the geometric background:

\[
G_{\mu\nu} = \eta_{\mu\nu} + tg_{\mu\nu}^{(1)} + t^2 g_{\mu\nu}^{(2)} + \ldots
\]

ii) \( S_0 = \frac{1}{2\pi h} \int_{\Sigma} d^2z \eta_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu \)

\[
X^\mu(z)X^\nu(w) \sim h\eta^{\mu\nu} \ln |z - w|^2
\]

logarithmic VOA
Nevertheless, if we neglect logarithms Maurer-Cartan equation

\[ Q\Phi + \frac{1}{2}\{\Phi, \Phi\} + \cdots = 0, \quad \Phi(G, B, \Phi) \]

reproduces Einstein equations \((H = dB)\):

\[
R_{\mu\nu} = -\frac{1}{4}H_{\mu\lambda\rho}H^{\lambda\rho}_{\nu} + 2\nabla_{\mu}\nabla_{\nu}\phi,
\]

\[
\nabla_{\mu}H^{\mu\nu\rho} - 2(\nabla_{\lambda}\phi)H^{\lambda\nu\rho} = 0,
\]

\[
4(\nabla_{\mu}\phi)^2 - 4\nabla_{\mu}\nabla^{\mu}\phi + R + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} = 0
\]

and their symmetries up to the second order in \(t\).

A.M.Z.

"Formal Maurer-Cartan Structures: from CFT to Classical Field Equations"
JHEP0712:098(2007)

We need another formulation:

\[
S = \frac{1}{2\pi h} \int_{\Sigma} d^2z \left( p_i \partial X^i + p_{\bar{i}} \partial X^{\bar{i}} -
\right.
\]

\[
\left. (g^{i\bar{j}} p_i p_{\bar{j}} + \mu^{i\bar{j}} p_i \partial X^{\bar{j}} + \mu^{\bar{i}j} p_{\bar{i}} \partial X^{j} + b_{ij} \partial X^i \partial X^j) \right) + \int_{\Sigma} R^{(2)} \hat{\phi}
\]

A.S. Losev, A.V. Marshakov, A.M.Z.

"On first order formalism in string theory"
Complication: \( g, b, \mu, \bar{\mu}, \hat{\phi} \longrightarrow G, B, \phi \) nonlinear transformation:

\[
G_{sk} = g_{ij} \bar{\mu}_s^i \mu^j_k + g_{sk} - b_{sk},
\]
\[
B_{sk} = g_{ij} \bar{\mu}_s^i \mu^j_k - g_{sk} - b_{sk},
\]
\[
G_{si} = -g_{ij} \bar{\mu}_s^j - g_{sj} \bar{\mu}_i^j,
\]
\[
G_{\bar{s}i} = -g_{sj} \mu^j_s - g_{ij} \mu^j_s,
\]
\[
B_{si} = g_{sj} \bar{\mu}_s^j - g_{ij} \bar{\mu}_s^j,
\]
\[
B_{\bar{s}i} = g_{ij} \mu^j_s - g_{sj} \mu^j_i,
\]
\[
\phi = \log \sqrt{g + \hat{\phi}}.
\]

Advantage:

\[
S_0 = \frac{1}{2\pi\hbar} \int_\Sigma d^2z \left( p_i \bar{\partial} X^i + p_i \partial X^{\bar{i}} \right)
\]

VOA:

\[
X^i(z)p_j(w) \sim \frac{h \delta^i_j}{z - w}, \quad X^{\bar{i}}(z)p_{\bar{j}}(w) \sim \frac{h \delta^{\bar{i}}_{\bar{j}}}{\bar{z} - \bar{w}}
\]
When $\mu, \bar{\mu}, b = 0$

$$G_{\bar{i\bar{j}}} = -B_{i\bar{j}} = g_{i\bar{j}}, \quad \phi = \hat{\phi} + \log \sqrt{g}$$

Resulting Einstein equations appear to be bilinear in $g_{i\bar{j}}$

$$\partial_i \partial_{\bar{k}} \Phi_0 = 0, \quad \partial_p d_{\bar{l}}^{\Phi_0} g_{\bar{i\bar{k}}} = 0, \quad \partial_p d_{\bar{l}}^{\Phi_0} g_{\bar{k}l} = 0,$$

$$2g_{\bar{r}\bar{l}} \partial_r \partial_{\bar{l}} g_{\bar{i\bar{k}}} - 2\partial_r g_{\bar{i}\bar{p}} \partial_{\bar{p}} g_{\bar{r}\bar{k}} - g_{\bar{i}\bar{l}} \partial_{\bar{l}} d_s^{\Phi_0} g_{\bar{s}\bar{k}} -$$

$$g_{\bar{r}\bar{k}} \partial_r d_{\bar{j}}^{\Phi_0} g_{\bar{j}\bar{i}} + \partial_r g_{\bar{i}\bar{k}} d_{\bar{j}}^{\Phi_0} g_{\bar{j}\bar{r}} + \partial_{\bar{p}} g_{\bar{k}\bar{i}} d_n^{\Phi_0} g_{\bar{n}\bar{p}} = 0,$$

where $d_{\bar{i}}^{\Phi_0} g_{\bar{i}\bar{j}} \equiv \partial_i g_{\bar{i}\bar{j}} - 2\partial_i \Phi_0 g_{\bar{i}\bar{j}}$ and $d_{\bar{i}}^{\Phi_0} g_{\bar{i}\bar{j}} \equiv \partial_i g_{\bar{j}\bar{i}} - 2\partial_i \Phi_0 g_{\bar{j}\bar{i}}$.

They are equivalent to:

$$Q\Phi + \frac{1}{2} \{\Phi, \Phi\} = 0, \quad [b_{-1}, [\bar{b}_{-1}, \Phi]] = g_{\bar{i}\bar{j}} p_i p_{\bar{j}}$$

at the order $h^1$.

Symmetries (holomorphic):

$$\Phi \longrightarrow \Phi + Q\Lambda + \{\Phi, \Lambda\} + \{\Lambda, \Phi\}$$

A.M.Z.

“Perturbed $\beta$-$\gamma$ Systems and Complex Geometry”,

Yang-Mills equations

\[ *d_A * F = 0, \quad F = dA + A \wedge A \]

\[ 0 \rightarrow \Omega^0 \overset{d}{\rightarrow} \Omega^1 \overset{d \ast d}{\rightarrow} \Omega^{D-1} \overset{d}{\rightarrow} \Omega^D \rightarrow 0 \]

Yang-Mills \( C_\infty \) algebra:

\[ 0 \rightarrow \mathcal{F}^0 \overset{Q_\eta}{\rightarrow} \mathcal{F}^1 \overset{Q_\eta}{\rightarrow} \mathcal{F}^2 \overset{Q_\eta}{\rightarrow} \mathcal{F}^3 \rightarrow 0 \]

\((\cdot, \cdot) : \mathcal{F}^i \otimes \mathcal{F}^j \rightarrow \mathcal{F}^{i+j}\)

\((\cdot, \cdot, \cdot) : \mathcal{F}^i \otimes \mathcal{F}^j \otimes \mathcal{F}^k \rightarrow \mathcal{F}^{i+j+k-1}\)

\[
\begin{array}{c|c|c|c|c|c}
 f_1 & v & A & V & a \\
\hline
 f_2 & vw & Aw & Vw & aw \\
 w & vB & (A, B) & B \wedge V & 0 \\
 B & vW & A \wedge W & 0 & 0 \\
 W & vb & 0 & 0 & 0 \\
 b & & & & & \\
\end{array}
\]

\[ (A, B) = (A \wedge *dB) - (B \wedge *dA) + d * (A \wedge B) \]

\[ (A, B, C) = A \wedge * (B \wedge C) - C \wedge *(A \wedge B) \]

\[ *d_A * F = 0 \iff Q_\eta A + (A, A) + (A, A, A) = 0 \]

See e.g. A.M.Z.
“Conformal Field Theory and Algebraic Structure of Gauge Theory”, arXiv:0812.1840
This algebra can be obtained from

\[ S = \frac{1}{2\pi \hbar} \int_D d^2z \partial X^\mu \bar{\partial} X^\nu \eta_{\mu\nu} + \int_{\partial D} A_\mu(X) dX^\mu \]

Complex \((\mathcal{F}^*, Q_\eta)\) is a subcomplex in the BRST complex \((C^*, Q_\eta)\) of open string. LZ operations \(\mu, \nu\) reproduce operations \((\cdot, \cdot)\) and \((\cdot, \cdot, \cdot)\) at the level \(h^0\).

In order to get rid of logarithms we need introduce first order formulation:

\[ S = \frac{1}{2\pi \hbar} \int_{H^+} d^2z (p_\mu \bar{\partial} X^\mu + \bar{p}_\mu \bar{\partial} X^\mu - \eta^{\mu\nu} p_\mu \bar{p}_\nu) + \int_{\mathbb{R}} dz (A_\mu(X) \partial X^\mu + B^\mu(X) p_\mu) \]

Boundary conditions:

\[ p_\mu |_{\mathbb{R}} = \bar{p}_\mu |_{\mathbb{R}}, \quad X^\mu |_{\mathbb{R}} = \bar{X}^\mu |_{\mathbb{R}} \]

VOA:

\[ X^\mu(z) p_\nu(w) \sim \frac{h \delta^\mu_\nu}{z - w} \]

Gaussian integration:

\[ A_\mu(X) \partial X^\mu + B^\mu(X) p_\mu \rightarrow A_\mu(X) dX^\mu, \]

where \( A_\mu(X) = A_\mu(X) + \eta_{\mu\nu} B^\nu(X) \)
Extended BRST operator of open string:

\[ \hat{Q}_\eta = Q_{X,p} + \eta^{\alpha\beta} \mu(a_\alpha, \{a_\beta, \cdot\}) \]

\[ F\hat{Q}_\eta = Q_\eta F \]

where \( Q_{X,p} \) is a BRST operator for \( X-p \) VOA, and \( a_\alpha = cp_\alpha \).

It is possible to deform \( \mu \) w.r.t. \( \eta^{\alpha\beta} \) in such a way that the new operation \( \mu^n \) will be homotopy commutative and associative w.r.t. \( \hat{Q}_\eta \). The complex \((\mathcal{F}^*, Q_\eta)\) lies in the kernel of \( L_0 \) (0th Virasoro mode of \( X-p \) VOA):

\[
\begin{align*}
\rho_u &= u(X), \quad \phi'_A = cA_\mu(X)\partial X^\mu, \\
\phi''_B &= c : B^\mu(X)p_\mu :, \\
\phi'_a &= \partial c a(X), \quad \psi'_V = c\partial c V_\mu(X)\partial X^\mu, \\
\psi''_W &= c\partial c : W^\mu(X)p_\mu :, \\
\psi_b &= c\partial^2 c b(X), \quad \chi_v = c\partial c\partial^2 c v(X).
\end{align*}
\]

Combining deformed \( \mu^n, n^n \) we find that they reproduce YM \( C_\infty \) algebra on \((\mathcal{F}^*, Q_\eta)\).

Relation to Courant/Dorfman algebroid:

\[
\{\phi'_A + \phi''_B, \phi'_A + \phi''_B\} = h(\phi'_{LB\bar{A}} - d_{iB}A + \phi''_{[B,\bar{B}]Lie})
\]

A.M.Z.

“\( \beta-\gamma \) systems and the deformations of BRST operator”, to appear
Chiral de Rham complex and Kodaira-Spencer Theory

**VOA (V):**

\[ X^i(z) p_j(w) \sim \frac{1}{z - w}, \quad \psi^i(z) \chi_j(w) \sim \frac{1}{z - w} \]

Chiral de Rham cohomology operator:

\[ Q = \frac{1}{2\pi i} \oint \psi^i p_i \]

Bilinear operation \( \{ a, b \} = \frac{(-1)^{|a|}}{2\pi i} \oint dz [Q, a](z)b \) satisfies Gerstenhaber algebra together with \( \mu. \) \( \{ \cdot, \cdot \} \) reproduces Schouten bracket on the operators \( f^{i_1 \ldots i_n}(x) \chi_{i_1} \ldots \chi_{i_n} \)

F. Malikov

“Lagrangian approach to Sheaves on Vertex Algebras”

Introducing antichiral part (\( \bar{V} \)): 

\[ \bar{X}^i(z) \bar{p}_j(w) \sim \frac{1}{\bar{z} - \bar{w}}, \quad \bar{\psi}^i(\bar{z}) \bar{\chi}_j(\bar{w}) \sim \frac{1}{\bar{z} - \bar{w}} \]

Let us consider elements of the form 

\[ \xi_{i_1 \ldots i_n}^{\bar{j}_1 \ldots \bar{j}_n}(X, \bar{X}) \chi_{i_1} \ldots \chi_{i_n} \psi^{\bar{j}_1} \ldots \psi^{\bar{j}_n} \in V \otimes \bar{V} \]

Then 

\[ \bar{Q} \mu + \{ \mu, \mu \} = 0, \quad \mu = \mu_j^i(X, \bar{X}) \chi_i \psi^{\bar{j}} \] coincides with Kodaira-Spencer equation.

Actually, most of Barannikov-Kontsevich formulas

S. Barannikov, M. Kontsevich

“Frobenius Manifolds and Formality of Lie algebras of Polyvector Fields”,
alg-geom/9710032

may be reproduced on VOA language.
Conclusions

• Physical interpretation of “higher homotopies” in Lian-Zuckerman construction.

• Construction of the field theory equations (Einstein, YM, Kodaira-Spencer) via the deformation theory of semi-infinite cohomology operator for certain VOAs.

• Algebraic approach to the study of $\beta$-function in $\sigma$-models.

• Relation of Courant/Dorfman algebroid and Yang-Mills $C_\infty$-algebra.