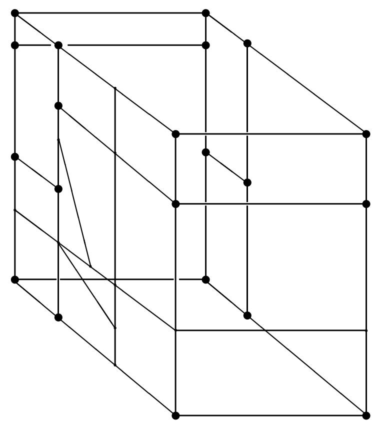
The A_∞ Matrad and the Polytopes KK



The Polytope $KK_{2,4}$

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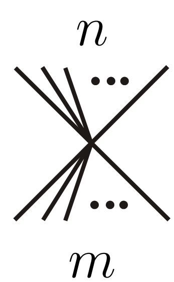
Goal: Specify the combinatorics of $KK_{n,m}$ in dimensions ≤ 3

- $\dim KK_{n,m} = n + m - 3$
- $KK_{n,1} = KK_{1,n} = K_n$ is Stasheff's associahedron

Strategy: Define the cellular boundary ∂ on top dim'l faces and extend inductively to lower dimensions.

- Matrad generators are double corollas

$\theta_m^n =$ 

thought of as operations in a PROP

$$M = \left\{ M_{n,m} = Hom\left(H^{\otimes m}, H^{\otimes n}\right)\right\}$$

- Data flows upward

Markl's Fraction Product

Monomials generated by the θ_m^n's have form

$$\alpha_x^y = \frac{\alpha_p^{y_1} \cdots \alpha_p^{y_q}}{\alpha_{x_1}^q \cdots \alpha_{x_p}^q},$$

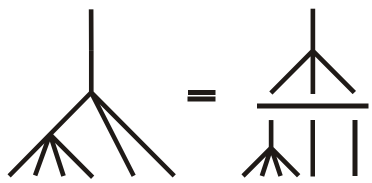
where

1. $x = \sum x_i$ and $y = \sum y_j$
2. # of outputs from each factor below is # of factors above
3. # of inputs to each factor above is # of factors below
4. The j^{th} output of the i^{th} factor below links to the i^{th} input of j^{th} factor above

- The operad of up-rooted Planar Rooted Trees (PRTs) is the free associative algebra

$$A = FA\Big(\quad , \quad , \quad , \quad \ldots \Big)$$

- Monomials in A are fraction products:



- Here $\partial\left(\theta_m^1\right) = \sum_{p=m-1} \alpha_p^1 \in A \cdot A$

  (we ignore signs)

- Similarly, in the operad of down-rooted PRTs

$$\partial\left(\theta_1^n\right) = \sum_{q=n-1} \alpha_1^q \in C \cdot C,$$

where $C = FA\left(\quad , \quad , \quad , \quad , \ldots \right)$

Define $\partial\left(\theta_m^n\right)$ by appropriately restricting the fraction product to a *matrad product.*

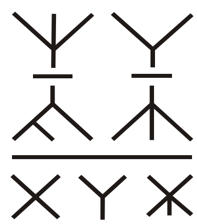
The Matrad Product in dimensions ≤ 3

The *lower and upper leaf sequences* of a monomial

$$\alpha_x^y = \frac{\alpha_p^{y_1} \cdots \alpha_p^{y_q}}{\alpha_{x_1}^q \cdots \alpha_{x_p}^q}$$

$$LLS\left(\alpha_x^y\right) = \left(x_1, \ldots, x_p\right)$$

$$ULS\left(\alpha_x^y\right) = \left(y_1, \ldots, y_q\right)$$

Example: For $\alpha_6^5 =$  ,

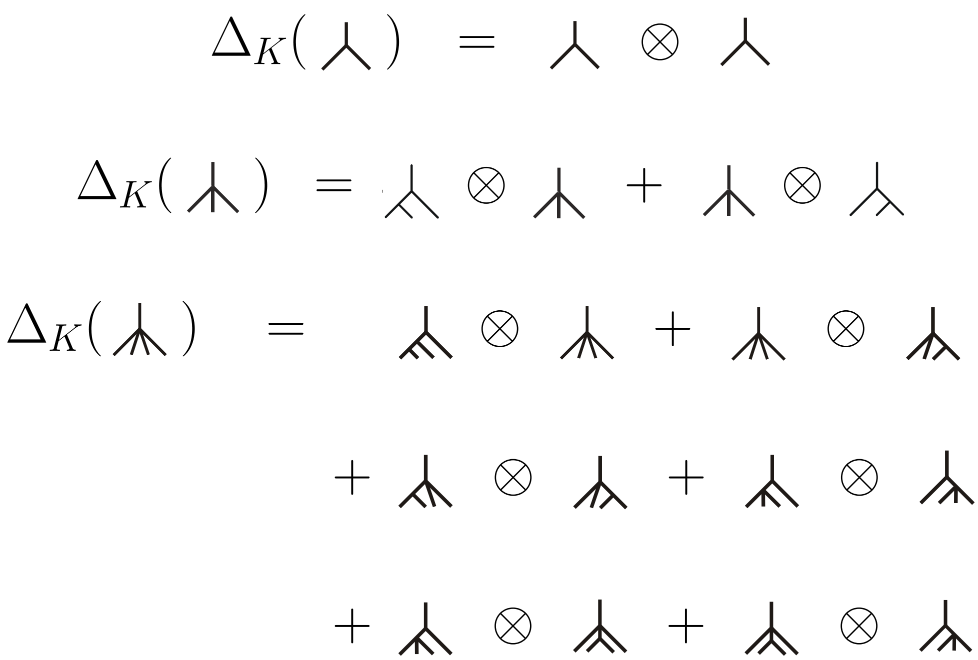
$$LLS\left(\alpha_6^5\right) = (2,1,3)\,; \; ULS\left(\alpha_6^5\right) = (3,2)\,.$$

The *lower and upper contact sequences* of α_6^5 are the lists of lower and upper leaf sequences

$$((2)\,,(2)\,,(2)) \;\; \text{and} \;\; ((2,1)\,,(3))\,.$$

S-U Diagonal on Associahedra

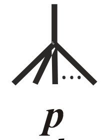
Define $\Delta_K : C_*(K_n) \to C_*(K_n) \otimes C_*(K_n)$ by



Define the *left-iterated diagonal* via

$$\Delta_K^{(0)} = \mathrm{id}$$

$$\Delta_K^{(k)} = \left(\Delta_K \otimes \mathrm{id}^{\otimes k-1}\right) \Delta_K^{(k-1)}$$

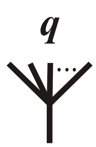
- View each component of $\Delta_K^{(k)}($ 

$)$ as an $(p-2)$-dim'l subcomplex of $K_p^{\times k+1}$

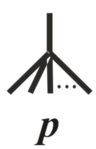
- A non-vanishing *matrad monomial*

$$\alpha_x^y = \frac{\alpha_p^{y_1} \cdots \alpha_p^{y_q}}{\alpha_{x_1}^q \cdots \alpha_{x_p}^q}$$

in dimension ≤ 3 satisfies

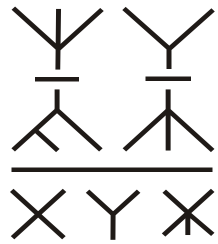
1. The lower contact sequence of α_x^y agrees with the list of ULS's of some component of $\Delta_K^{(p-1)}($ 

$)$

2. The upper contact sequence of α_x^y agrees with the list of LLS of some component of $\Delta_K^{(q-1)}($ 

$)$

Example: In

$$\alpha_6^5 \quad = \quad$$  ,

the lower contact sequence $((2),(2),(2))$ is the list of ULS's of

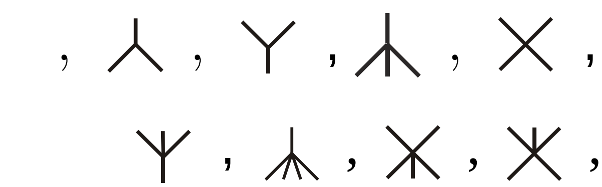
$$\mathsf{Y} \otimes \mathsf{Y} \otimes \mathsf{Y} \;=\; \Delta_K^{(2)}(\mathsf{Y})$$

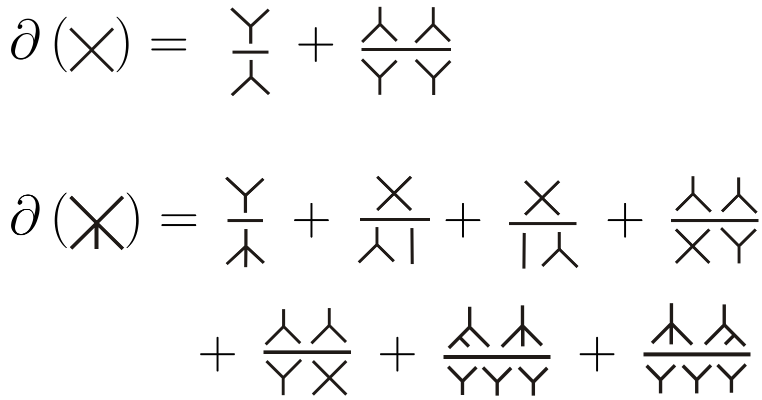
and the upper contact sequence $((2,1),(3))$ is the list of LLS's of

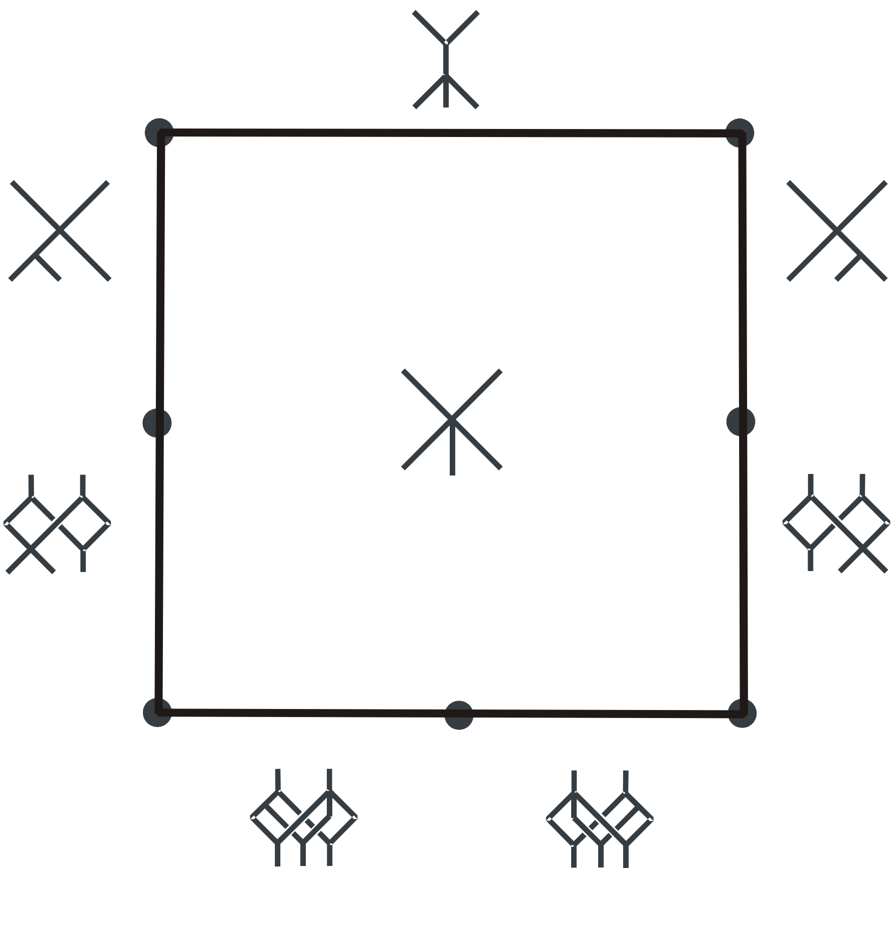
 $\otimes$  in $\Delta_K^{(1)}($  $)$.

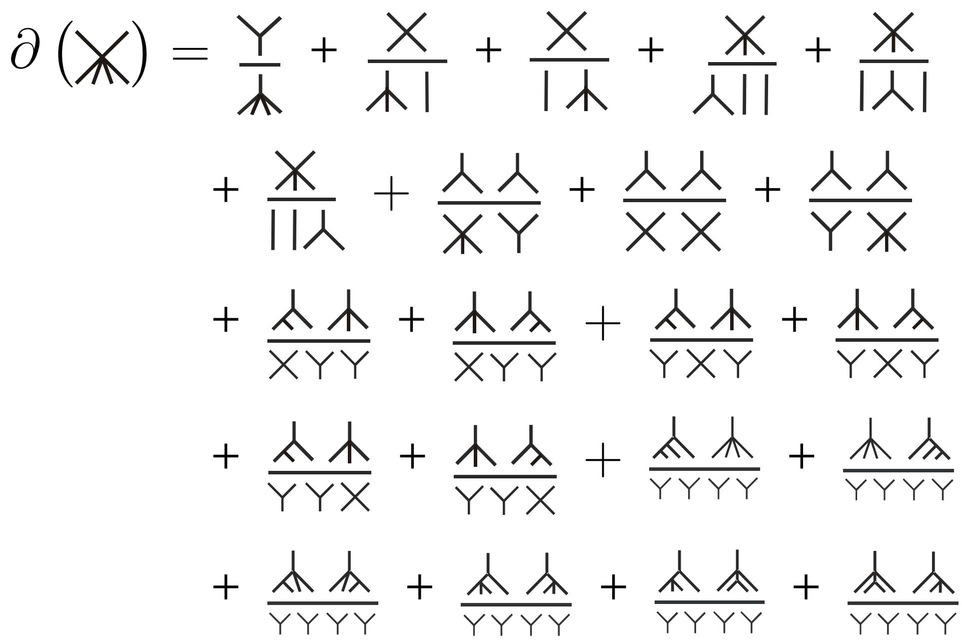
For $m+n \leq 6$, define

$$\partial\left(\theta_m^n\right) \;=\; \sum_{p+q=m+n-1} \alpha_p^q \in B \cdot B,$$

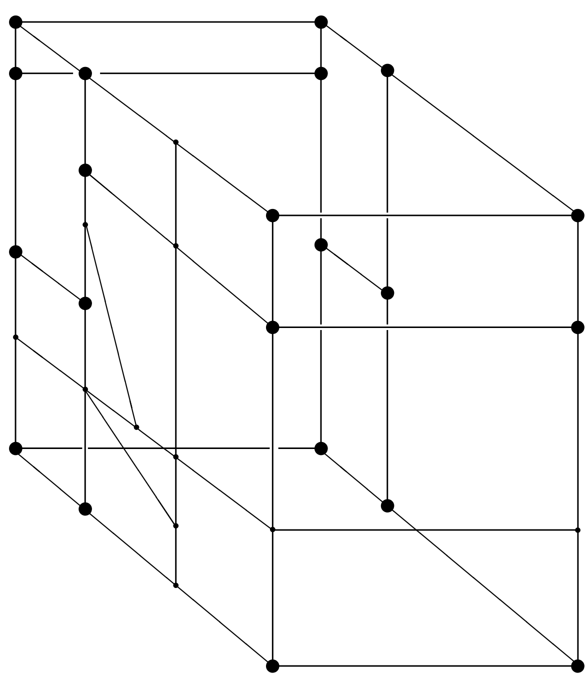
where $B = FA\Big($  $\Big)$



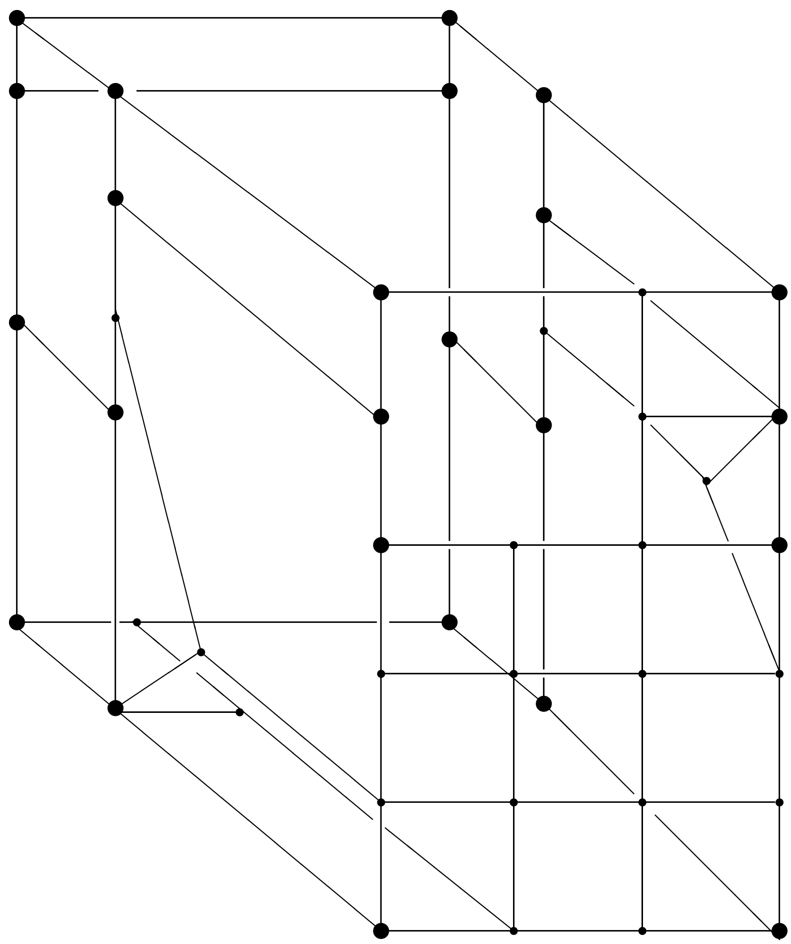




- Note iterated Δ_K in “numerator/denominator”



$KK_{3,3}$:



A_∞-bialgebras

- The *bialgebra matrad* $\mathcal{H}_\infty$ is realized by $C_*(KK)$
- $End(TH)$ is canonically a matrad
- A map of matrads
$$\mathcal{H}_\infty \to End(TH)$$
imposes an A_∞-*bialgebras structure* on an R-module H.
- An A_∞-*bialgebra* is an algebra over $\mathcal{H}_\infty$

An alternative...

The Biderivative

Choose arbitrary

$$\omega_m^n \in M_{n,m} = Hom\left(H^{\otimes m}, H^{\otimes n}\right)$$

and consider $\omega = \sum_{m,n\geq 1} \omega_m^n$

The "biderivative" $d_\omega : M \to \mathbf{M}$ induces a non-bilinear operation

$$\circledcirc : M \times M \overset{d_\bullet \times d_\bullet}{\longrightarrow} \mathbf{M} \times \mathbf{M} \overset{\Upsilon}{\longrightarrow} \mathbf{M} \overset{proj}{\to} M$$

Alternative Definition:

(H, ω) *is an* $\underline{A_\infty\text{-bialgebra}}$ *if* $\omega \circledcirc \omega = 0$

Construct d_ω as follows:

- Linearly extend $d = \omega_1^1$ to $(H^{\otimes p})^{\otimes q}$

- Freely extend the map
$$\sum_{j \geq 1} \omega_1^j : H \to T^a H$$
as a derivation

- Cofreely extend the map
$$\sum_{i \geq 1} \omega_i^1 : T^c H \to H$$
as a coderivation

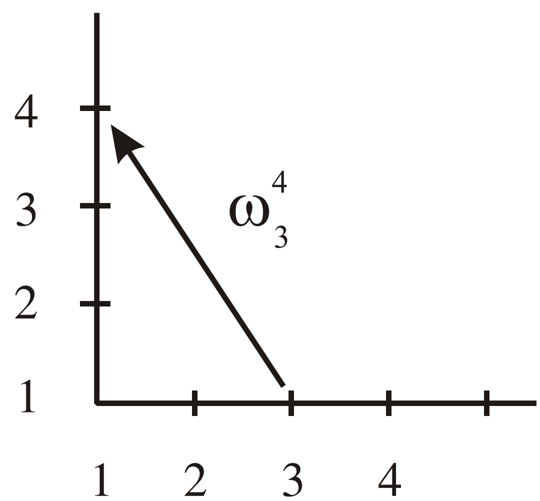
- Freely extend the map
$$\sum_{j > 1} \omega_i^j : H^{\otimes i} \to T^a H$$
as a Δ_P-derivation for each i

- Cofreely extend the map
$$\sum_{i > 1} \omega_i^j : T^c H \to H^{\otimes j}$$
as a Δ_P-coderivation for each j

To picture this, make the identification

$$\left(H^{\otimes p}\right)^{\otimes q} \leftrightarrow (p, q) \in \mathbb{N}^2$$

and represent $\omega_p^q : H^{\otimes p} \longrightarrow H^{\otimes q}$ as a "transgressive" arrow $(p, 1) \longrightarrow (1, q)$:



- Represent components A and B of the extensions above as arrows in $\mathbb{N}^2$

- When the terminal point of B is the initial point of A, the (restricted) fraction product γ is given by

$$\gamma\left(A \otimes B\right) = A \circ \sigma_{p,q} \circ B,$$

where $\sigma_{p,q} : \left(H^{\otimes p}\right)^{\otimes q} \xrightarrow{\approx} \left(H^{\otimes q}\right)^{\otimes p}$ is the canonical permutation of tensor factors

- Define $\omega \circledcirc \omega = \sum_{A,B \in d_\omega} \gamma\left(A \otimes B\right)$

- Summands of $\omega \circledcirc \omega$ have one of two types:

1. $\gamma(\omega_j^k \otimes \underbrace{(\mathbf{1} \cdots \cdot \omega_i^1 \cdots \mathbf{1})}_{j})$ and vise versa
2. $\gamma\left(A_1 \cdots A_t \otimes B_1 \cdots B_s\right)$ with $s, t \geq 2$

   (sequences of arrows)

- When $\omega \circledcirc \omega = 0,$ there is a relation involving certain "transgressive" products $\gamma\,(A \otimes B)$ from $(p, 1)$ to $(1, q)$ for each p and q

Example: When $\omega_2^2 = 0$ there is the relation