Algebraic Birkhoff decomposition and renormalization of multiple zeta values *

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*joint work with Bin Zhang

0. MZVs with positive arguments

Multiple zeta values are defined to be the evaluation of the multiple complex variable function

$$\zeta(\vec{s}) = \zeta(s_1, \cdots, s_k) = \sum_{n_1 > \cdots > n_k > 0} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}},$$

at positive integers $s_1, \cdots, s_k$ with $s_1 > 1$. (Euler, Hoffman, Zagier, ...).

Double shuffle relations:

$$\begin{array}{ll} \text{Shuffle :} & \zeta(\vec{s} ш \vec{t}) = \zeta(\vec{s})\zeta(\vec{t}), \\ \text{Quasi} - \text{shuffle :} & \zeta(\vec{s} * \vec{t}) = \zeta(\vec{s})\zeta(\vec{t}). \end{array}$$

Question: What about MZVs with nonpositive arguments?

1. Non-positive MZVs by analytic continuation

One variable case: $\zeta(z)$ has analytic continuation to the whole complex plane with a simple pole at $z = 1$. So $\zeta(k), k \leq 0$, are defined. Further,

$$\zeta(k) = (-1)^k \frac{B_{-k+1}}{-k+1}.$$

Multiple variable case: In two variable case considered by Atkinson (1949) and in general by Goncharov, Zhao, Arakawa-Kaneko, and Akiyama-Egami-Tanigawa (around 2000).

Theorem. $\zeta(s_1, \cdots, s_k)$ can be meromorphically continued to $\mathbb{C}^k$ with singularities on the subvarieties

$$\begin{aligned} s_1 &= 1; \\ s_1 + s_2 &= 2, 1, 0, -2, -4, \cdots; \text{ and} \\ \sum_{i=1}^{j} s_i &\in \mathbb{Z}_{\leq j}, \ (3 \leq j \leq k). \end{aligned}$$

Behavior near the singularities: Simple poles. For example, near $(0,0)$

$$\zeta(s_1, s_2) = \frac{5s_1 + 4s_2}{12(s_1 + s_2)} + R_2(s_1, s_2)$$

where $R_2(s_1, s_2)$ is an entire function near $(0,0)$ with $R_2(0,0) = 0$.

Problem: Most special values of $\zeta(s_1, \cdots, s_k)$, $k > 1$, at nonpositive integers are not defined.

MZVs as directional limits (Akiyama, Egami and Tanigawa): They proposed several definitions of multiple zeta functions at $(s_1, \cdots, s_k)$ where s_i are all non-positive.

$$\begin{aligned}
\zeta(s_1, \cdots, s_k) &= \lim_{r_1 \to s_1} \cdots \lim_{r_k \to s_k} \zeta(r_1, \cdots, r_k), \\
\zeta(s_1, \cdots, s_k) &= \lim_{r_k \to s_k} \cdots \lim_{r_1 \to s_1} \zeta(r_1, \cdots, r_k), \\
\zeta(s_1, \cdots, s_k) &= \lim_{r \to 0} \zeta(s_1 + r, \cdots, s_k + r).
\end{aligned}$$

1. The definitions do not agree with each other. For $\zeta(r_1, r_2)$ at $(r_1, r_2) = (0, 0)$, the proposed values are $5/12, 1/3, 3/8$ respectively. In fact, by letting (r_1, r_2) approach to $(0, 0)$ along different paths, one can get any values, as well as the infinity, to be the limit.
2. The values do not satisfy either the shuffle of quasi-shuffle relation.

2. Non-positive MZVs by renormalization: examples

The concept of regularization and renormalization of MZVs has been used by Zagier and several other authors to deal with degenerated cases of the double shuffle relation in order to obtain (conjecturally) all algebraic relations among MZVs.

More recently, Dominique Manchon and Sylvie Paycha (2005-6) considered renormalization of multiple zeta value from the point of view of Chen integrals and Chen sums of symbols using a similar renormalization approach á la Connes and Kreimer.

Our approach to deal with non-positive multiple zeta values,

- in one variable case, use the generating function of Bernoulli numbers;
- in multiple variable case, use the renormalization procedure in QFT, in the Hopf algebra framework of Connes-Kreimer.

One variable case: $\frac{\varepsilon}{e^{\varepsilon}-1}=\sum_{k\geq 0} B_k \frac{\varepsilon^k}{k!}$ gives

$$\frac{e^{\varepsilon}}{1-e^{\varepsilon}}=-\frac{1}{\varepsilon}\frac{-\varepsilon}{e^{-\varepsilon}-1}=-\frac{1}{\varepsilon}+\sum_{k\geq 0}\zeta(-k)\frac{\varepsilon^k}{k!}$$

since $\zeta(-k)=(-1)^k\frac{B_{k+1}}{k+1}$ for $k\geq 0$.

Regularized zeta values: For $s\leq 0$,

$$Z(s;\varepsilon):=\sum_{n\geq 1}\frac{e^{n\varepsilon}}{n^s}=\frac{d^{-s}}{d\varepsilon}\left(\frac{e^{\varepsilon}}{1-e^{\varepsilon}}\right)=\frac{d^{-s}}{d\varepsilon}\left(-\frac{1}{\varepsilon}+\sum_{k\geq 0}\zeta(-k)\frac{\varepsilon^k}{k!}\right).$$

Define fp $\left(\sum_{n\geq N} a_n\varepsilon^n\right)$ to be the power series part of the Laurent series. Then for the "renormalized value", we have (Euler)

$$\text{fp}\, Z(s,\varepsilon)\big|_{\varepsilon=0}=\zeta(s).$$

Two variable case:

The regularization of the formal multiple zeta value

$$\zeta(s_1, s_2) = \sum_{n_1 > n_2 > 0} \frac{1}{n_1^{s_1} n_2^{s_2}}$$

is

$$Z(s_1, s_2; \varepsilon) = \sum_{n_1 > n_2 > 0} \frac{e^{n_1} e^{n_2}}{n_1^{s_1} n_2^{s_2}}.$$

For example,

$$Z(0, 0; \varepsilon) = \sum_{n_1 > n_2 > 0} e^{n_1 \varepsilon} e^{n_2 \varepsilon} = \frac{e^{\varepsilon}}{1 - e^{\varepsilon}} \frac{e^{2\varepsilon}}{1 - e^{2\varepsilon}}.$$

To “evaluate” $\zeta(0, 0)$, first naively take the finite part:

$$\mathrm{fp}\Big(\sum_{n_1 > n_2 > 0} e^{n_1 \varepsilon} e^{n_2 \varepsilon} \Big)\Big|_{\varepsilon = 0} = 11/24.$$

This value does not satisfy the quasi-shuffle (stuffle) relation:

$$1/4 = \zeta(0)\zeta(0) \neq 2\zeta(0, 0) + \zeta(0) = 5/12.$$

A principle in a renormalization procedure in QFT is that if a divergent Feynman integral contains a component integral that is already divergent, then the divergency of the component integral has to be removed before removing the divergency of the integral itself.

Adapting this principle of QFT to the proposed definition of $\zeta(0,0)$, we found that $\zeta(0,0)$ should be defined to be

$$\mathrm{fp}\Big(\sum_{n_1>n_2>0} e^{n_1\varepsilon}e^{n_2\varepsilon} - \sum_{n_2>0} e^{n_2\varepsilon}\underbrace{\big(\sum_{n_1>0} e^{n_1\varepsilon} - \mathrm{fp}(\sum_{n_1>0} e^{n_1\varepsilon})\big)}_{\text{sub}-\text{divergence}}\Big)\Big|_{\varepsilon=0} = \frac{3}{8}$$

This value indeed satisfies the quasi-shuffle relation

$$\zeta(0)\zeta(0) = 2\zeta(0,0) + \zeta(0).$$

How does this work in general?

3. Rota-Baxter algebras:

A Rota-Baxter operator or Baxter operator on an algebra A is a linear map $P : A \to A$ such that

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \ \forall\, x, y \in A.$$

Examples: Integration: $A = \mathbf{Cont}(\mathbb{R})$ (ring of continuous functions on $\mathbb{R}$).

$$P : A \to A, P[f](x) := \int_0^x f(t)dt.$$

Then P is a weight 0 Rota-Baxter operator by the integration by parts formula:

$$\int_0^x F(t)G'(t)dt = F(x)G(x) - \int_0^x F'(t)G(t)dt.$$

Iterations of P give iterated integrals in multiple zeta values and yields the shuffle product.

Summation: On a suitable class of functions, define

$$P(f)(x) := \sum_{n\geq 1} f(x+n).$$

Then P is a Rota-Baxter operator of weight 1. Its iterations give multiple zeta values and multiple polylogarithms and yields the quasi-shuffle product, called mixable shuffle product in free commutative Rota-Baxter algebras.

QFT dimensional regularization: Let $R = \mathbb{C}[t^{-1}, t]]$ be the ring of Laurent series $\sum_{n=-k}^{\infty} a_n t^n,\ \ k \geq 0$. Define

$$P(\sum_{n=-k}^{\infty} a_n t^n) = \sum_{n=-k}^{-1} a_n t^n.$$

Then P is a Rota-Baxter operator of weight -1.

4. Algebraic Birkhoff decomposition.

Let $\mathcal{H}$ be a connected filtered Hopf algebra, R a commutative Rota-Baxter algebra (with $\lambda = -1$) with idempotent Rota-Baxter operator P.

Theorem (Algebraic Birkhoff decomposition) Any algebra homomorphism $\phi : \mathcal{H} \to R$ has a unique decomposition into algebra homomorphisms

$$\phi = \phi_-^{-1} \star \phi_+, \quad \begin{cases} \phi_- : \mathcal{H} \to \mathbb{C} + P(R) \ \text{(counter term)} \\ \phi_+ : \mathcal{H} \to \mathbb{C} + (\mathrm{id} - P)(R) \ \text{(renormalization)} \end{cases}$$

Connes-Kreimer (2000) for $R = \mathbb{C}[\varepsilon^{-1}, \varepsilon]]$ in terms of loop decomposition.
Ebrahimi-Fard, Guo, Kreimer (2004) for general R in terms of Atkinson's decompositions in Rota-Baxter algebras.

◇ In QFT renormalization (Dim-Reg scheme), we take the triple $(\mathcal{H}_{\mathsf{FG}}, R, \phi)$:

- Hopf algebra $\mathcal{H}_{\mathsf{FG}}$ of Feynman graphs;
- $R = \mathbb{C}[\varepsilon^{-1}, \varepsilon]]$ of Laurent series, with the pole part projection P;
- $\phi : \mathcal{H}_{\mathsf{FG}} \to R$ from dimensional regularized Feynman rule.

$$\begin{tikzcd}
\mathcal{H}_{\mathsf{FG}} \arrow[r, "\text{Feynman rule}"] \arrow[d, "\phi_+\big|_{\varepsilon=0}"'] \arrow[dr, "\text{regularized Feynman rule } \phi"] & \text{Feynman integrals}(=\infty!) \\
\text{renormalized integrals} \in \mathbb{C} & \text{regularized integrals} \in \mathbb{C}[\varepsilon^{-1}, \varepsilon]]
\end{tikzcd}$$

◇ To define MZVs for not necessarily positive integers, henceforth called **renormalized MZVs**, we construct a triple $(\mathcal{H}_{\mathrm{MZV}}, \mathcal{R}_{\mathrm{MZV}}, \phi_{\mathrm{MZV}})$ of

- Hopf algebra $\mathcal{H}_{\mathrm{MZV}}$ of general MZVs;
- $\mathcal{R}_{\mathrm{MZV}} = \mathbb{C}[T][\varepsilon^{-1}, \varepsilon]]$ of (log) Laurent series, with the pole part projection P;
- $\phi_{\mathrm{MZV}} : \mathcal{H}_{\mathrm{MZV}} \to \mathcal{R}_{\mathrm{MZV}}$ of the regularized MZV rule.

Then apply the Algebraic Birkhoff Decomposition

$$\phi_{\mathrm{MZV}} = \phi_{\mathrm{MZV},-} \star \phi_{\mathrm{MZV},+}.$$

$$\begin{tikzcd}
\mathcal{H}_{\mathrm{MZV}} \arrow[rr, "\text{MZV rule}"] \arrow[d, "\phi_{\mathrm{MZV},+}\big|_{\varepsilon=0}"'] \arrow[drr, "\text{regularized MZV rule } \phi_{\mathrm{MZV}}"'] & & \text{formal MZVs}(=\infty!) \\
\text{renormalized MZVs} \in \mathbb{C} & & \text{regularized MZVs} \in \mathcal{R}_{\mathrm{MZV}}
\end{tikzcd}$$

5. The construction of the MZV triple $(\mathcal{H}_{\mathsf{MZV}}, \mathcal{R}_{\mathsf{MZV}}, \phi_{\mathsf{MZV}})$.

The Hopf algebra of general MZVs: Commutative semigroup

$$\mathfrak{M} = \{f_{s,r} \mid (s,r) \in \mathbb{Z} \times \mathbb{R}_{>0}\}$$

with the multiplication $f_{s,r} f_{s',r'} = f_{s+s',r+r'}$. Let $A_{\mathfrak{M}} = \mathbb{C}\,\mathfrak{M}$ be the commutative (semi-)group algebra. Then

$$\mathcal{H}_{\mathsf{MZV}} := \sum_{n \geq 0} A_{\mathfrak{M}}^{\otimes n} = \sum_{n \geq 0} \mathbb{C}\,\mathfrak{M}^n,$$

with the quasi-shuffle (=mixable shuffle) product and deconcatenation coproduct, is a filtered connected Hopf algebra.

Quasi-shuffle product: Let A be a commutative algebra. For $\mathfrak{a} = a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}$ and $\mathfrak{b} = b_1 \otimes \cdots \otimes b_n \in A^{\otimes n}$, define

$$\begin{aligned} \mathfrak{a} * \mathfrak{b} \;=\; & a_1 \otimes \big((a_2 \otimes \cdots \otimes a_m) * \mathfrak{b}\big) + b_1 \otimes \big(\mathfrak{a} * (b_2 \otimes \cdots \otimes b_n)\big) \\ & + (a_1 b_1) \otimes \big((a_2 \otimes \cdots \otimes a_m) * (b_2 \otimes \cdots \otimes b_n)\big) \end{aligned}$$

with the convention that if $m = 1$ then $a_2 \otimes \cdots \otimes a_m$ is the identity.

Deconcatenation coproduct: $\Delta(a_1 \otimes \cdots \otimes a_m)$

$$= (a_1 \otimes \cdots a_m) \bigotimes 1 + (a_1 \otimes \cdots a_{m-1}) \bigotimes a_m + \cdots + 1 \bigotimes (a_1 \otimes \cdots \otimes a_m)$$

Laurent series of regularized general MZVs:
Let $\mathcal{R}_{\text{MZV}} = \mathbb{C}[T]\{\{\varepsilon\}\}[\varepsilon^{-1}]$ and $R_{\text{MZV}} := \mathbb{C}\{\{\varepsilon\}\}[\varepsilon^{-1}]$. Let $\mathcal{P}$ and P be the projections to the pole parts. Then both $(\mathcal{R}, \mathcal{P})$ and (R, P) are Rota-Baxter algebras.

Regularized MZV **rule** ϕ_{MZV}: For $(\vec{s}, \vec{r}) \in \mathbb{Z}^k \times \mathbb{R}^k_{>0}$, define

$$\phi_{\text{MZV}}(f_{\vec{s},\vec{r}}) := Z(\vec{s}, \vec{r}; \varepsilon) = \sum_{n_1 > \cdots > n_k > 0} \frac{e^{n_1 r_1 \varepsilon} \cdots e^{n_k r_k \varepsilon}}{n_1^{s_1} \cdots n_k^{s_k}} \left(= \text{Li}_{s_1, \cdots, s_k}(e^{r_1 \varepsilon}, \cdots, e^{r_k \varepsilon})\right)$$

where

$$\text{Li}_{s_1, \cdots, s_k}(z_1, \cdots, z_k) = \sum_{n_1 > \cdots > n_k > 0} \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1^{s_1} \cdots n_k^{s_k}}$$

is the multiple polylogarithms.

Theorem. i) $Z(\vec{s}, \vec{r}; \varepsilon)$ has unique expansion in

$$\mathbb{C}\{\{\varepsilon\}\}[\varepsilon^{-1}][\ln \varepsilon] \cong \mathbb{C}\{\{\varepsilon\}\}[\varepsilon^{-1}][T] \hookrightarrow \mathcal{R}_{\text{MZV}} := \mathbb{C}[T]\{\{\varepsilon\}\}[\varepsilon^{-1}].$$

ii) For $\vec{s} \in \mathbb{Z}^k_{\leq 0}$, $Z(\vec{s}, \vec{r}; \varepsilon)$ has a unique Laurent series expansion in $R_{\text{MZV}} := \mathbb{C}\{\{\varepsilon\}\}[\varepsilon^{-1}]$.
iii) The map $\phi_{\text{MZV}} := Z : \mathcal{H}_{\text{MZV}} \to \mathcal{R}_{\text{MZV}}$, $f_{\vec{s},\vec{r}} \mapsto Z(\vec{s}, \vec{r}; \varepsilon)$ is an algebra homomorphism.

6. Renormalization of regularized MZVs:

Let $\mathcal{R}_+ = \mathbb{C}[T]\{\{\varepsilon\}\}, \mathcal{R}_- = \mathbb{C}[T][\varepsilon^{-1}]$.

Algebraic Birkhoff for $Z = \phi_{\mathsf{MZV}}$: There is a unique decomposition

$$Z = Z_-^{-1} \star Z_+$$

of Z into algebra homomorphisms $Z_\pm : \mathcal{H}_{\mathsf{MZV}} \to \mathcal{R}_\pm$. Define

$$\zeta(\vec{s}, \vec{r}) = Z_+(\vec{s}, \vec{r}; \varepsilon)|_{\varepsilon=0}.$$

Corollary of Algebraic Birkhoff: $\zeta(\vec{s}, \vec{r})$ satisfies the quasi-shuffle relation.

Proof: $\zeta : \mathcal{H}_{\mathsf{MZV}} \to \mathbb{C}$ is the composition of algebra homomorphisms $Z_+ : \mathcal{H}_{\mathsf{MZV}} \to \mathcal{R}_+$ and $eva_{\varepsilon=0} : \mathcal{R}_+ \to \mathbb{C}[T]$, so is still an algebra homomorphism. Since product in $\mathcal{H}_{\mathsf{MZV}}$ satisfies quasi-shuffle relation, so does their images.

Renormalized general MZVs For $\vec{s} \in \mathbb{Z}^k_{>0} \cup \mathbb{Z}^k_{\leq 0}$, define

$$\bar{\zeta}(\vec{s}) = \lim_{\delta \to 0^+} \zeta(\vec{s}, |\vec{s}| + \delta),$$

where, for $\vec{s} = (s_1, \cdots, s_k)$ and $\delta \in \mathbb{R}_{>0}$, we denote $|\vec{s}| = (|s_1|, \cdots, |s_k|)$ and $|\vec{s}| + \delta = (|s_1| + \delta, \cdots, |s_k| + \delta)$. These $\bar{\zeta}(\vec{s})$ are called the (renormalized) **general multiple zeta values (gMZVs)** of the multiple zeta function $\zeta(u_1, \cdots, u_k)$ at $\vec{s}$.

Theorem: $\bar{\zeta}(\vec{s})$ are well-defined and satisfy the quasi-shuffle relation.

In the following table, the element in row s_1 and column s_2 is $\bar{\zeta}(-s_1, -s_2)$, $1 \leq s_1 \leq 7,\ 1 \leq s_2 \leq 8$.

$\frac{1}{288}$	$-\frac{1}{240}$	$\frac{83}{64512}$	$\frac{1}{504}$	$-\frac{3925}{2239488}$	$-\frac{1}{480}$	$\frac{342884347}{99656663040}$
$-\frac{1}{240}$	0	$\frac{1}{504}$	$-\frac{319}{437400}$	$-\frac{1}{480}$	$\frac{2494519}{1362493440}$	$\frac{1}{264}$
$-\frac{71}{35840}$	$\frac{1}{504}$	$\frac{1}{28800}$	$-\frac{1}{480}$	$\frac{114139507}{139519328256}$	$\frac{1}{264}$	$-\frac{313042283533}{93600000000000}$
$\frac{1}{504}$	$\frac{319}{437400}$	$-\frac{1}{480}$	0	$\frac{1}{264}$	$-\frac{41796929201}{26873437500000}$	$-\frac{691}{65520}$
$\frac{32659}{15676416}$	$-\frac{1}{480}$	$-\frac{21991341}{25836912640}$	$\frac{1}{264}$	$\frac{1}{127008}$	$-\frac{691}{65520}$	$\frac{26194796926873}{5884626295848960}$
$-\frac{1}{480}$	$-\frac{2494519}{1362493440}$	$\frac{1}{264}$	$\frac{41796929201}{26873437500000}$	$-\frac{691}{65520}$	0	$\frac{1}{24}$
$-\frac{75497471}{19931332608}$	$\frac{1}{264}$	$\frac{316292283533}{93600000000000}$	$-\frac{691}{65520}$	$-\frac{36808933898915}{8238476814188544}$	$\frac{1}{24}$	$\frac{1}{115200}$
$\frac{1}{264}$	$\frac{16608667097}{2879296875000}$	$-\frac{691}{65520}$	$-\frac{4607695}{491051484}$	$\frac{1}{24}$	$\frac{63967403428993199}{3561322226607185040}$	$-\frac{3617}{16320}$

Equivalent definitions and compatibility with known MZVs:

Convergent MZVs:

$$\bar{\zeta}(\vec{s}) = \zeta(\vec{s}, \vec{s})$$

and $\bar{\zeta}(\vec{s}) = \zeta(\vec{s})$.

Positive MZVs: For s_i all positive,

$$\bar{\zeta}(\vec{s}) = \zeta(\vec{s}, \vec{s})$$

and agrees with the regularization of Zagier et al.

Negative MZVs: For s_i all negative,

$$\bar{\zeta}(\vec{s}) = \zeta(\vec{s}, -\vec{s})$$

and agree with the values by analytic continuation. They are $\zeta(s)$ for $s \leq 0$ and $\zeta(s_1, s_2)$ with $s_1, s_2 \leq 0$ and $2 \not| s_1 + s_2$.

Further directions:

- MZVs with arbitrary arguments;
- Shuffle relation of general MZVs;
- Double shuffles and (rational) associators;
- Other regularizations;
- Other multiple zeta values.

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