Solving First Order Ordinary Differential Equations Using Lie Symmetries

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Abstract

We would like to use Lie symmetries of first order ordinary differential equations to solve them. Specifically, we will examine one-parameter Lie groups associated with first order ODEs. We will then use this information to study one of the standard methods taught in a beginning course in ODEs.

1 Introduction

The methods presented in this paper come from the first two chapters of Peter Hydon's book *Symmetry Methods for Differential Equations: A Beginner's Guide*. The first step in using Lie symmetries to solve a first order ODE is to understand exactly what is meant by symmetry, and more specifically Lie symmetries associated with a first order ODE. We will discuss these definitions, and from them, we arrive at a symmetry condition for first order ODEs. Then we will look at how to use Lie symmetries to solve a first order ODE. Finally, we will use Lie Symmetries to solve some first order ODEs and to study one of the so-called "standard" methods that is taught in a beginning course in ordinary differential equations.

2 Symmetries

We start by looking at symmetries of rigid objects. For example, a rigid square has eight discrete symmetries that consist of rotations and reflections. On the other hand, a unit circle has an infinite number of continuous symmetries that can be described by $\Gamma_\epsilon(x, y) = (\hat{x}, \hat{y}) = (x\cos\epsilon - y\sin\epsilon, x\cos\epsilon + y\sin\epsilon)$, $\epsilon\mathbb{R}$. While discrete symmetries are interesting and useful, we will focus on continuous symmetries. So symmetries are diffeomorphisms that map $\mathbb{R}^2$ to $\diagdown^2$. These diffeomorphisms preserve the structure of the object. The set of diffeomorphims $\{\Gamma_\epsilon\}$ form a group under composition.

The group of symmetries $\{\Gamma_\epsilon\}$ is called a **one-parameter Lie group** if the following hold:

1. $\epsilon = 0$ is the trivial symmetry,
2. $\Gamma_\epsilon\Gamma_\delta = \Gamma_{\epsilon+\delta}$.

Then the set of symmetries, $\Gamma_\epsilon(x, y) = (x\cos\epsilon - y\sin\epsilon, x\cos\epsilon + y\sin\epsilon)$, for the circle form a one-parameter Lie group, which we will sometimes refer to as Lie symmetries.

3 Symmetry Condition

Our discussion will be restricted to solving first-order ordinary differential equations, so we will look at symmetries $(x, y) \to (\hat{x}, \hat{y})$ and require that $\frac{d\hat{y}}{d\hat{x}} = \omega(\hat{y}, \hat{x})$ when $\frac{dx}{ds} = \omega(x, y)$. We get the symmetry condition:

$$\begin{aligned}\frac{d\hat{y}}{d\hat{x}} &= \frac{D_x\hat{y}}{D_x\hat{x}} \\ &= \frac{\hat{y}_x + y'\hat{y}_y}{\hat{x}_x + y'\hat{x}_y} \\ &= \frac{\hat{y}_x + \omega(x, y)\hat{y}_y}{\hat{x}_x + \omega(x, y)\hat{x}_y} = \omega(\hat{x}, \hat{y}).\end{aligned}$$

Note that D_x refers to the total derivative with repsect to x, $\hat{y}_x$ refers to the partial derivative of $\hat{y}$ with respect to x, and y' refers to $\frac{dy}{dx}$. Remember that we want our symmetries to be a diffeomorphisms, so we require $\hat{x}_x\hat{y}_y = \hat{x}_y\hat{y}_x \neq 0$.

We can check to see if a transformation on (x, y) is a symmetry of a given differential equation by checking that the transformation satisfies the symmetry condition:

$$\frac{\hat{y}_x + \omega(x, y)\hat{y}_y}{\hat{x}_x + \omega(x, y)\hat{x}_y} = \omega(\hat{x}, \hat{y}) \tag{1}$$

Example 3.1. *Let* $\frac{dy}{dx} = \frac{y^3 + x^2y - y - x}{xy^2 + x^3 + y - x}$. *If we look at a graph of the solution field, it appears that rotations about the origin are symmetries. We can* *indeed check that the set of rotations* $(\hat{x}, \hat{y}) = (x\cos(\epsilon) - y\sin(\epsilon), x\sin(\epsilon) + y\cos(\epsilon))$ *are symmetries of the differential equation. These rotations form a one-parameter Lie group of symmetries, which we will just call Lie symmetries.*

We check this by substituting into the symmetry condition (1). See the Maple code below.

```
> w:=(x,y)->(y^3+x^2*y-y-x)/(x*y^2+x^3+y-x):
> xhat:=x*cos(epsilon)-y*sin(epsilon):
> yhat:=x*sin(epsilon)+y*cos(epsilon):
> what:=simplify(w(xhat,yhat));
```

$$what := \frac{-x\sin(\varepsilon) + y\sin(\varepsilon) + \sin(\varepsilon)\,x^3 + \sin(\varepsilon)\,y^2\,x - x\cos(\varepsilon) - y\cos(\varepsilon) + x^2\,y\cos(\varepsilon) + y^3\cos(\varepsilon)}{x\sin(\varepsilon) + y\sin(\varepsilon) - \sin(\varepsilon)\,x^2\,y - \sin(\varepsilon)\,y^3 - x\cos(\varepsilon) + y\cos(\varepsilon) + x^3\cos(\varepsilon) + y^2\,x\cos(\varepsilon)}$$

```
> symm_cond:=(diff(yhat,x)+w(x,y)*diff(yhat,y))/
> (diff(xhat,x)+w(x,y)*diff(xhat,y)):
> symm:=simplify(symm_cond);
```

$$symm := \frac{-x\sin(\varepsilon) + y\sin(\varepsilon) + \sin(\varepsilon)\,x^3 + \sin(\varepsilon)\,y^2\,x - x\cos(\varepsilon) - y\cos(\varepsilon) + x^2\,y\cos(\varepsilon) + y^3\cos(\varepsilon)}{x\sin(\varepsilon) + y\sin(\varepsilon) - \sin(\varepsilon)\,x^2\,y - \sin(\varepsilon)\,y^3 - x\cos(\varepsilon) + y\cos(\varepsilon) + x^3\cos(\varepsilon) + y^2\,x\cos(\varepsilon)}$$

```
> evalb(what=symm);
```

true

In the above code, we have defined $w(x, y)$ *as the right-hand side of the differential equation. We then define* $\hat{x}$ *and* $\hat{y}$*, where* $\Gamma(\hat{x}, \hat{y})$ *is the symmetry we are trying to check. Substitute* $\hat{x}$ *and* $\hat{y}$ *into* w*. Finally, we compute the* *symmetry condition and use the boolean evaluation operator to check that*

$$\frac{\hat{y}_x + \omega(x,y)\hat{y}_y}{\hat{x}_x + \omega(x,y)\hat{x}_y} = \omega(\hat{x},\hat{y}).$$

It turns out if we can find a one-parameter Lie group of symmetries for a differential equation (ordinary or partial), then we can use the group to find the general solutions. In first-order ODEs, finding the Lie symmetries may be just as difficult as solving the equation itself. However, in higher order ODEs and PDEs, finding the symmetries is significantly easier than solving the differential equation.

Again, we will just focus on first-order ODEs in this paper to develop some of the techniques needed. We would like to be able to take a first-order ODE and use the symmetries to solve the equation by separation of variables. Separation of variables will allow us to integrate to find the solution. We need a few more concepts before we can do this. Specifically, we will need the concept of canonical coordinates.

4 Actions of the Lie Group of Symmetries

The goal is to be able to solve any differential equation of the form

$$\frac{dy}{dx} = \omega(x,y). \tag{2}$$

Notice that if we have a solution to (2), given by $y = f(x)$, then the Lie symmetries map the solution to some curve $\hat{y} = \tilde{f}(\hat{x})$. In other words, a point $(x, f(x))$

on the solution is mapped to $(\hat{x}, \tilde{f}(\hat{x}))$. Notice that the solution curve is invariant iff $f = \tilde{f}$ and that the Lie symmetry is trivial iff every solution curve is left invariant under the action of the symmetry.

The one-parameter Lie group of symmetries for (2) depends on some parameter ϵ. We saw above how the Lie group transforms solution curves. The orbit of the group through a point (a, b) is the set of points that (a, b) can be mapped to by the group. Notice that for $\epsilon = 0$, we get the original point. Typically the orbit of a point is a smooth curve. For example, the orbits for the rotational group given above are circles, except for (0,0). The point (0,0) is a fixed point, i.e., it is mapped to itself. Notice that any point along the orbit is just mapped to another point on the orbit. This tells us that orbits are invariant under the action of the Lie group of symmetries.

Supppos e we have a one-parameter Lie group of symmetries generated by $(\hat{x}, \hat{y})$. Take an orbit through a point (x, y) and look at the tangent vector to the orbit at (x, y). The tangent vector is given by

$$(\xi(x, y), \eta(x, y)) = (\frac{d\hat{x}}{d\epsilon}|_{\epsilon=0}, \frac{d\hat{y}}{d\epsilon}|_{\epsilon=0}). \tag{3}$$

This tangent vector is called the infinitesimal generator. The point was arbitrary, so we can define an infinitesimal generator at each point on the orbit. The set of these infinitesimal generators forms a smooth vector field. We say the point (x, y) is fixed only if $(\xi(x, y), \eta(x, y)) = (0, 0)$.

Suppose C is some curve in the (x, y) plane. If an orbit crosses C transversely at some point, then the point in question can be mapped to points not on C. This means that C is not an invariant curve. We end up with the result that C is invariant if and only if the tangent to C at each point (a, b) is parallel to $(\xi(a, b), \eta(a, b))$. We can write this condition as

$$Q(x, y, y') = \eta(x, y) - y'\xi(x, y). \tag{4}$$

Then the tangent to C at (a, b) is parallel to $(\xi(a, b), \eta(a, b))$ if and only if

$$Q(x, y, y') = 0 \text{ on C} \tag{5}$$

The equation (4) is called the **characteristic**. We will rewrite $Q(x, y, y') = Q(x, y, \omega(x, y)) = \tilde{Q}(x, y) = \eta(x, y) - \omega(x, y)\xi(x, y)$. This is the **reduced characteristic**. A solution curve is invariant if and only if the reduced characteristic is zero on the solution curve. Recall that a Lie symmetry is trivial if it leaves all points fixed. Then we know that the Lie symmetries are trivial if $\tilde{Q}$ is identically zero.

We will use the infinitesimal generators rather than the Lie symmetries to solve first-order ODEs. However, we get the infinitesimal generators from nontrivial Lie symmetries. The reduced characteristic gives us a quick way to verify if the Lie symmetries are trivial. We will see that to solve first-order ODEs, we need nontrivial Lie symmetries.

5 Canonical Coordinates

We want to find canonical coordinates that allow us to solve the given first-order ODE by using integration. First look back to the symmetry condition. Look at the differential equation (2). Suppose $(\hat{x}, \hat{y}) = (x, y + \epsilon)$ generates a one-parameter Lie group of symmetries for (2). Plug this into the symmetry condition (1). See the Maple code below:

```
> xhat:=x:

> yhat:=y+epsilon:
> symm_cond:=(diff(yhat,x)+omega*diff(yhat,

> y))/(diff(xhat,x)+omega*diff(xhat,y));
```

$$symm\_cond := \omega$$

Notice that the symmetry condition becomes $\omega(x, y) = \omega(x, y + \epsilon)$. If we differentiate with respect to ϵ at $\epsilon = 0$, then $\omega y(x, y) = 0$, which implies that $\frac{dy}{dx} = \omega(x)$. This ODE can be solved by separation of variables. This will be true for any first-order differential equation that has translational symmetry.

Our goal then is to find canonical canonical coordinates so that the Lie symmetries become translations. Then we can rewrite the differential equation in terms of canonical coordinates and solve by separation of variables. Once we find a solution, we change the coordinates back to get a solution for the original ODE.

Again, let $(\hat{x}, \hat{y}) = (x, y + \epsilon)$. Then the tangent vector at any point on any orbit for this group looks like $(\xi(x, y), \eta(x, y)) = (0, 1)$. We want new coordinates $(r, s) = (r(x, y), s(x, y))$, such that $(r(\hat{x}, \hat{y}), s(\hat{x}, \hat{y})) = (\hat{r}, \hat{s}) = (r, s + \epsilon)$. Then the tangent vector at (r, s) is defined by $(\frac{d\hat{r}}{d\epsilon}|_{\epsilon} = 0, \frac{d\hat{s}}{d\epsilon}|_{\epsilon=0}) = (0, 1)$.

Find the tangent vector at (r, s) by using the chain rule to get:

$$\xi(x, y)r_x + \eta(x, y)r_y = 0, \tag{6}$$

$$\xi(x, y)s_x + \eta(x, y)s_y = 1. \tag{7}$$

As usual, we want the change of coordinates to be locally invertible (local to (x, y)), so $r_x s_y - r_y s_x \neq 0$. Then the pair $(r, s) = (r(x, y), s(x, y))$ subject to (6) and (7) is called **canonical coordinates**. Note that we cannot define canonical coordinates at a fixed point. Remember that at a fixed point, $(\xi(x, y), \eta(x, y)) = (0, 0)$. If this is the case, then (7) has no solution.

We can find r by solving the differential equation $\frac{dy}{dx} = \frac{\eta(x,y)}{\xi(x,y)} = c = r$, where c is a constant and $\xi(x, y)$ is not zero. This differential equation sis typically easier to solve than the original one. We then use r to find s. Solve $r(x, y)$ for y. Then

$$s(r, x) = \left(\int \frac{dx}{\xi(x, y(r, x))} \right) |_{r=r(x,y)}.$$

If instead, $\xi(x, y) = 0$, then set $r = x$ and

$$s = \left(\int \frac{dy}{\eta(r, y)} \right) |_{r=x},$$

provided that $\eta(x, y)$ is nonzero. Later, we will give some examples that involve calculating canonical coordinates.

6 Solving ODEs with Canonical Coordinates

In the previous section, we used the generators of a one-parameter Lie group of symmetries for a differential equation of the form (2) to find canonical coordinates. The canonical coordinates allow us to rewrite the differential equation in such a way that it can be solved using separation of variables.

Suppose that we have nontrivial Lie symmetries, $(\hat{x}, \hat{y})$ for a differential equation of the form (2) such that (r, s) are canonical coordinates. Recall that the Lie symmetries are nonzero if and only if

$$\eta(x, y) \neq \omega(x, y)\xi(x, y).$$

Rewrite the ODE as

$$\frac{ds}{dr} = \frac{s_x + \omega(x, y)s_y}{r_x + \omega(x, y)r_y} \tag{8}$$

Once rewriting the ODE as (8), plug in r and s for x and y to get $\frac{ds}{dr}$ equal to a function of r and s. However, (r, s) is a pair of canonical coordinates. This means that the ODE in terms of r and s has translational symmetry and reduces down to

$$\frac{ds}{dr} = \Omega(r), \tag{9}$$

for some function Ω. Then the general solution to (9) becomes $s - \int \Omega(r)dr = c$, where c is a constant. To get the general solution for (2) just plug x and y back in for r and s.

Example 6.1. *We want to solve the differential equation:*

$$\frac{dy}{dx} = xy^2 - \frac{2y}{x} - \frac{1}{x^3}, \ x \neq 0 \tag{10}$$

This differential equation has the nontrivial Lie symmetries $(\hat{x}, \hat{y}) = (e^{\epsilon}x, e^{-2\epsilon}y)$*.*

Then $(\xi(x,y), \eta(x,y)) = (x, -2y)$*. We can find* r *by looking at*

$$\frac{dy}{dx} = \frac{-2y}{x} = c.$$

Performing separation of variables gives us $r = x^2y$*. Then* $y = rx^{-2}$ *and* $s = \int \frac{dx}{x}$*. Therefore* $(r, s) = (x^2y, \ln|x|)$ *are canonical coordinates. Rewrite the differential equation in terms of* r *and* s*:*

$$\frac{ds}{dr} = \frac{s_x + \omega(x,y)s_y}{r_x + \omega(x,y)r_y} \tag{11}$$

$$\frac{ds}{dr} = \frac{\frac{1}{x} + (xy^2 - \frac{2y}{x} - \frac{1}{x^3})(0)}{2xy + (xy^2 - \frac{2y}{x} - \frac{1}{x^3})x^3} \tag{12}$$

$$\frac{ds}{dr} = \frac{1}{2x^2y + x^4y^2 - 2x^2y - 1} \tag{13}$$

$$\frac{ds}{dr} = \frac{1}{x^4y^2 - 1} \tag{14}$$

$$\frac{ds}{dr} = \frac{1}{r^2 - 1} \tag{15}$$

Using separation of variables and partial fractions, we obtain

$$s = \frac{1}{2}\ln\left(\frac{r-1}{r+1}\right) + c,$$

for some constant c*. If we put* x *and* y *into the solution and solve for* y*, we end up with* $y = \frac{x^2+k}{x^2(k-x^2)}$*, for some constant* k*.*

Look at the solution and suppose $k \neq 0$*. If we apply the symmetry, then the* *solution is transformed to*

$$e^{-2\epsilon}y = \frac{(e^{\epsilon}x)^2 + k}{(e^{\epsilon}x)^2(k - (e^{\epsilon}x)^2)}.$$

This reduces to

$$y = \frac{ke^{-2\epsilon} + x^2}{x^2(ke^{-2\epsilon} - x^2)}.$$

Then for $k \neq 0$, we get a family of nonsymmetric solutions for the differential equation. Notice that for $k = 0$, the solution is mapped to itself. This is also the case as k approaches infinity or negative infinity. These solutions are called symmetric solutions. We will revisit the concept of symmetric solutions and families of nonsymmetric solutions.

Notice that if the Lie symmetry had been trivial, then $\frac{dy}{dx} = \frac{\eta(x,y)}{\xi(x,y)} = \frac{\omega(x,y)\xi(x,y)}{\xi(x,y)}$ reduces down to the original ODE, which does not help to solve for y.

7 First-Order ODEs of Homogeneous Type

The use of symmetries to find canonical coordinates leads to methods usable for some classes of ODEs. Some of these methods are taught in a first course in ODEs and are refered to as "standard methods." One class of ODEs studied in a first course are first-order ODEs of homeogeneous type.

First-order ODEs of homogeneous type are of the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \tag{16}$$

Students are taught in a first course in ODEs to make the substitution $z(x) = \frac{y(x)}{x}$, and set $y(x) = z(x)x$. Then $y' = xz' + z = f(z)$. Solving $xz' + z = f(z)$ for z' gives us the separable equation $z' = \frac{f(z)-z}{x}$. Solve for $z(x)$ to get $y = xz(x)$. We will see how to get this from using canonical coordinates.

Every ODE of the form given in (16) has $(\hat{x}, \hat{y}) = (e^{\epsilon}x, e^{\epsilon}y)$ as a one-parameter group of Lie symmetries. Notice that these symmetries are scalings of the differential equation, such that for $\epsilon = 0$, $(\hat{x}, \hat{y}) = (x, y)$. Use these generators to find canonical coordinates. The tangent vector at (x, y) is given by $(\xi(x, y), \eta(x, y)) = (x, y)$. Then we find r by solving the differential equation $\frac{dy}{dx} = \frac{y}{x} = c$ to get $r = \frac{y}{x}$. Solve for s by integrating $\frac{dx}{x}$ to get $\ln|x|$. The canonical coordinates for (16) are $(r, s) = (\frac{y}{x}, \ln|x|)$.

The next step is to rewrite the differential equation in terms of r and s:

$$
\begin{aligned}
\frac{ds}{dr} &= \frac{s_x + \omega(x,y)s_y}{r_x + \omega(x,y)r_y} && (17)\\
&= \frac{\frac{1}{x}}{-\frac{y}{x^2} + F\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)} && (18)\\
&= \frac{1}{-\frac{y}{x} + F\left(\frac{y}{x}\right))} && (19)\\
&= \frac{1}{F(r) - r}. && (20)
\end{aligned}
$$

For $F(r) \neq r$, we can solve this by separation of variables:

$$\frac{ds}{dr} = \frac{1}{F(r) - r} \tag{21}$$

$$ds = \frac{dr}{F(r) - r} \tag{22}$$

$$\ln|x| = \int \frac{dr}{F(r) - r} + c. \tag{23}$$

Then plug $\frac{y}{x}$ in for r to get the solution to the original ODE. Notice that this gives us exactly the method taught in ODEs.

If $F(r) = r$, then the symmetry is trivial and the general solution is $r = c$ or $y = cx$, where c is some constant.

The method we are taught for dealing with first-order ODEs of homogeneous type comes directly from the Lie symmetries of the ODE. We will conclude this section with an explicit example.

Example 7.1. *Solve the differential equation $xy + y^2 - x^2 \frac{dy}{dx} = 0$. Rewrite the ODE:*

$$x^2 \frac{dy}{dx} = xy + y^2 \tag{24}$$

$$\frac{dy}{dx} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 \tag{25}$$

We know that $(r, s) = (\frac{y}{x}, \ln|x|)$ is a pair of canonical coordinates for this dif-

ferential equation. Then we write the ODE in terms of r and s as follows

$$\begin{aligned}
\frac{ds}{dr} &= \frac{\frac{1}{x}}{-\frac{x}{y^2} + \left(\frac{y}{x} + \left(\frac{y}{x}\right)^2\right)\frac{1}{x}} && (26)\\
&= \frac{1}{-\frac{y}{x} + \frac{y}{x} + \left(\frac{y}{x}\right)^2} && (27)\\
&= \frac{1}{\left(\frac{y}{x}\right)^2} && (28)\\
&= \frac{1}{r^2} && (29)
\end{aligned}$$

This equation is easily solved.

$$\begin{aligned}
ds &= r^{-2}dr && (30)\\
\int ds &= \int r^{-2}dr && (31)\\
s &= \frac{-1}{r} + c, \textit{ for some constant } c && (32)\\
\ln|x| &= \frac{-x}{y} + c && (33)\\
\ln|x| - c &= \frac{-x}{y} && (34)\\
y &= \frac{x}{c - \ln|x|}. && (35)
\end{aligned}$$

If we had solved this problem using the method taught in a first course in ODEs, we would see that the method does agree with the symmetry method used in the example.

8 Conclusion

Thus far, we have restricted our discussion to first-order ordinary differential equations. Unfortunately, in the first order case, oftentimes finding the symmetries is as difficult as solving the ODE. However, in higher order ODEs and

PDEs, finding the symmetries is significantly easier. We can then use the symmetries to solve the differential equation.

In higher order ODEs, we use the symmetries to reduce the order of the differential equation. For PDEs, methods for finding symmetric solutions are well understood. However, general methods to find nonsymmetric families of solutions are not known. Finding nonsymmetric families o solutions to PDEs is an open area of research.

References

[1] Edwards, C. Henry and Penney, David E. *Differential Equations: Computing and Modeling*. Upper Saddle River, NJ: Prentice Hall, 2000.

[2] Hydon, Peter E. *Symmetry Methods for Differential Equations: A Beginner's Guide*. Cambridge, UK: Cambridge University Press, 2000.