Crash Course in Group Actions

Katie Thompson

June 4, 2007

1 Groups

Definition 1.1. *A **binary operation** on a set S is a function that sends two elements in S to a single element in S.*

Let S be a set with a binary operation. The element produced by the binary operation on $a, b \in S$ is denoted by ab.

When $a, b \in S$ and $ab \in S$, we call S **closed** under the operation.

Example 1.2. *Let S be the set of 2×2 matrices. Addition is a binary operation on S. The addition of two matrices in S will produce a matrix in S.*

Definition 1.3. *A set G with a binary operation is a **group** if it satisfies the following properties:*

1. *There exists $e \in G$ such that $eg = ge = g$ for all $g \in G$. (identity).*
2. *For each $g \in G$ there exists an element g^{-1} such that $gg^{-1} = g^{-1}g = e$. (inverses).*
3. *The binary operation is associative. So for any $x, y, z \in G$ we have $(xy)z = x(yz)$.*

Definition 1.4. *A **subgroup** H of G is a subset of G that is also a group.*

A subgroup inherits the group operation, so it will inherit associativity. You need only check that H contains the identity, is closed under the group operation, and is closed under inverses. If H is a subgroup of G we denote it by $H \leq G$.

Definition 1.5. *A group G is called **abelian** if $ab = ba$ for all $a, b \in G$.*

2 Examples of Groups

Example 2.1. *$\mathbb{R}^n$ forms a group under vector addition.*

Example 2.2. *The set of invertible 2×2 matrices with real entries forms a group under multiplication. This group is denoted by $GL(2, \mathbb{R})$ or by $GL_2(\mathbb{R})$.*

Example 2.3. *The set of invertible* $n \times n$ *matrices with complex entries forms a group under multiplication and is denoted by* $GL_n(\mathbb{C})$.

Example 2.4. *The set* $SO(n) = \{A \in GL_n(\mathbb{R}) | A^t A = I, det(A) = 1\}$ *forms a group under multiplication. This is called the special orthogonal group. For* $n = 2$*, these matrices are rotations in the plane.*

The above groups are all examples of Lie groups. A Lie group is a group that is also a differentiable manifold. A Lie group also has the property that the group operations are smooth. We may discuss this more later.

With Lie groups, we talk about dimension rather than order. The dimension of a real Lie group is the number of independent real parameters that define the group.

Example 2.5. *The Lie group* $\mathbb{R}^n$ *has dimension* n*. Each vector contains* n *entries that are independent of one another.*

Question 2.6. *Try to determine the dimensions of* $GL_2(\mathbb{R})$*,* $GL_n(\mathbb{R})$*, and* $SO(2)$*.*

Example 2.7. *The symmetric group* S_3 *is the group of all permutations on three symbols. A permutation rearranges the elements of an ordered set so that the set is in a one-to-one correspondence with itself. This is an example of a finite group. The group* S_3 *consists of 6 (or* $3!$*) permutations.*

Question 2.8. *Notice that* $SO(n)$ *is a subgroup of* $GL_n(\mathbb{R})$*. Find two more examples of subgroups of* $GL_n(\mathbb{R})$*.*

Question 2.9. *Find two more examples of finite groups.*

Question 2.10. *Let* $A, B \in SO(2)$*. Verify that* $AB \in SO(2)$*.*

Question 2.11. *Let* $A \in SO(2)$*. Verify that* $A^{-1} \in SO(2)$*.*

Definition 2.12. *Let* G *and* $\overline{G}$ *be groups. A* ***group homomorphism*** $\phi : G \to \overline{G}$ *is a function that preserves the group operation. In other words,* $\phi(xy) = \phi(x)\phi(y)$ *for all* $x, y \in G$*.*

Question 2.13. *Show that the determinant function,* $det : GL_2(\mathbb{R}) \to \mathbb{R}^\times$ *is a group homomorphism.*

Definition 2.14. *A subgroup* $H \leq G$ *is* ***normal*** *if* $\forall h \in H$ *and* $\forall g \in G$*, then* $ghg^{-1} \in H$*.*

Example 2.15. *The group of invertible* $n \times n$ *matrices with determinant one (*$SL_n(\mathbb{R})$*) is a normal subgroup of* $GL_n(\mathbb{R})$*.*

Definition 2.16. *Let* $H \leq G$*. A* ***coset*** *is a set of the form* $gH = \{gh | h \in H\}$*. In particular, this is called a left coset.*

Example 2.17. *Any subgroup* H *is a coset because it can be written as* eH*, where* e *is the group identity.*

In general, cosets are not subgroups.

Definition 2.18. *Let H be a subgroup of G. The set of cosets of H in G is denoted by G/H.*

Theorem 2.19. *If H is a normal subgroup of G, then G/H forms a group called the **quotient group**.*

3 Group Actions

Definition 3.1. *Let G be a group and let S be a set. We can define an **action of G on S**. This is a rule for taking $g \in G$, $s \in S$ and assigning them to an element of S. The map $G \times S \to S$ must satisfy the following:*

1. *$es = s$ where e is the identity element in G and $s \in S$.*
2. *$(gg')s = g(g's)$ for all $g, g' \in G, s \in S$. (associative).*

*We say that G acts on S or G operates on S. Notice that we have defined a left operation, i.e. the group operates on the left. The set S is sometimes referred to as a **G-set**.*

We can also define a group action in terms of permutations.

Definition 3.2. *A **permutation** is a bijective map from a set S to itself.*

For a group G and a set S, we can define a group action as a homomorphism $\phi : G \to Symm(S)$, where $Symm(S)$ is the group of permutations on S.

Example 3.3. *The easiest example of a group action is a group acting on itself by left multiplication. So we could say that $GL_n(\mathbb{R})$ acts on $GL_n(\mathbb{R})$ by left multiplication. Notice that the above properties are satisfied.*

Example 3.4. *Let $G = SO(2)$ and let S be the set of points in the plane. G acts on S by rotating points about the origin.*

Question 3.5. *Describe an action of S_3 on the vertices of an equilateral triangle.*

Question 3.6. *Describe an action of S_3 on the vertices of a regular hexagon. (The answer is not unique.)*

Definition 3.7. *The **orbit** of a point in $s \in S$ is defined as the set $O_s = orb_G(s) = \{s' \in S | s' = gs \text{ for some } g \in G\}$.*

Example 3.8. *The orbit of the origin under the action of $SO(2)$ is the origin. The orbit of any other point in the plane under this action will be a circle. (See the figure at the top of the next page.)*

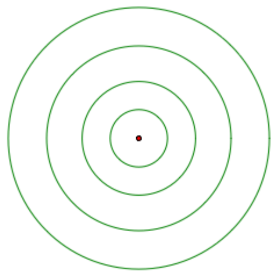

Figure 1: Orbits for $SO(2)$ acting on the plane

Definition 3.9. *An action is* ***transitive*** *if* $\exists s \in S$ *such that* $O_s = S$.

Notice that S_3 acting on the vertices of a triangle by all permutations is a transitive action.

Question 3.10. *Show that if* $\exists s \in S$ *such that* $O_s = S$*, then* $O_s = S\ \forall s \in S$.

Definition 3.11. *The* ***stabilizer*** *of an element* $s \in S$ *is defined as the set* $G_s = \{g \in G | gs = s\}$*. The stabilizer is sometimes referred to as the isotropy group of a point.*

Example 3.12. *Consider* $SO(2)$ *acting on the plane. The stabilizer of a point not at the origin is the* 2×2 *identity matrix. The stabilizer of the origin is the entire group.*

You should notice that the orbit of an element is a subset of S, and the stabilizer of an element is a subgroup of G.

Definition 3.13. *An action is called* ***free*** *if* $G_s = \{e\}\ \forall s \in S$.

Consider translations on the plane. For any point, the only translation that stabilizes the point is the identity. So translation on the plane is a free action.

Another example of a free action is the group of rotations on the plane without the origin. If we include the origin, then the action is no longer free because the stabilizer of the origin is the whole group.

Definition 3.14. *An action is called* ***faithful****, or* ***effective****, if*

$$\bigcap_{s \in S} G_s = \{e\}.$$

In other words, the identity is the only element that fixes everything.

Definition 3.15. *Let* $H = \bigcap_{s \in S} G_s$*.* H *is called the* ***global isotropy subgroup****, or the* ***global stabilizer****.*

Theorem 3.16. *The global isotropy subgroup is a normal subgroup.*

Theorem 3.17. *Let G be a group acting on a set S, and let H be the global isotropy subgroup. Suppose the action is nonfaithful. Then the group G/H defines an equivalent faithful action.*

Theorem 3.18 (Orbit-Stabilizer Theorem for Finite Groups)**.** *Let G be a finite group acting on a set S. Then $|G| = |O_s||G_s|\ \forall s \in S$.*

There is a version of this theorem for Lie groups.

Theorem 3.19 (Orbit-Stabilizer Theorem for Lie Groups)**.** *Let G be a Lie group acting on a set S. Then $dim(G) = dim(O_s) + dim(G_s)\ \forall s \in S$.*

You should have observed that the dimension of $SO(2)$ is 1. Recall that the orbit of a point (not at the origin) is a circle. Then the orbit of that point has dimension 1. The identity is the only element that stabilizes the point, so the dimension of the stabilizer of the point is 0. You see then that the orbit-stabilizer theorem holds.

Question 3.20. *Consider S_3 acting on the vertices of a triangle. Find the orbit and the stabilizer of one of the vertices.*

Question 3.21. *Consider S_3 acting on the vertices of a hexagon. Find the orbit and the stabilizer of one of the vertices.*

Question 3.22. *Let G act on S. Let $s \in S$. Verify that G_s is a subgroup of G. (In other words, show that G_s is closed under the group operation. Also show that G_s is closed under inverses.)*

Question 3.23. *Refer back to your answers for questions 2.8 and 2.9. Describe a group action using the examples you came up with. Check the orbit-stabilizer theorem for these examples.*

4 Invariant Functions

Definition 4.1. *Let G be a group acting on a set S. A function $f : S \to \mathbb{R}$ is **invariant** if $f(gs) = f(s)\ \forall g \in G$.*

Example 4.2. *For $SO(2)$ acting on the plane, the radius function $r = \sqrt{x^2 + y^2}$ is invariant. Any function of the radius r is also invariant.*

Example 4.3. *The polynomial invariants under the action of S_3 on (x_1, x_2, x_3) are called the **symmetric functions**. There is a theorem that states that the symmetric functions can be written in terms of the elementary symmetric functions. The elementary symmetric functions for S_3 are given by:*

$$x_1 + x_2 + x_3 \tag{1}$$

$$x_1x_2 + x_1x_3 + x_2x_3 \tag{2}$$

$$x_1x_2x_3 \tag{3}$$

Question 4.4. *Try to come up with some invariant functions for the group actions you defined in question 3.23.*

We will discuss some methods for finding invariants later.

References

[1] Artin, Michael. *Algebra*. Upper Saddle River, NJ: Prentice Hall, Inc., 1991.

[2] Gallian, Joseph A. *Contemporary Abstract Algebra*. Houghton Mifflin College Div.; Fifth Edition, 2001.

[3] Olver, Peter J. *Classical Invariant Theory*. Cambridge, UK: Cambridge University Press, 1999.