

LAGRANGIAN RELAXATION TECHNIQUES FOR LARGE SCALE OPTIMIZATION

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Thought for the Day

In fact, the great watershed in optimization isn't between *linearity* and *nonlinearity*, but *convexity* and *nonconvexity*.

— R.T. Rockafellar (SIAM Review 1993)

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Lagrangian relaxation: 1

- Consider the following *primal* nonlinear program (P)

$$\begin{array}{ll} \max & f(x) \\ \text{s.t.} & g(x) \leq 0 \\ & h(x) = 0 \\ & x \in X \end{array}$$

- Problem (P) is a nonconvex or mixed integer problem. Want a good upper bound on its optimal value.
- Problem (P) is *easy*, but the set X has a *decomposable* structure.
- In either case, $g(x) \leq 0$ and $h(x) = 0$ are *complicating constraints* and maximizing $f(x)$ over the set X is easy.
- Consider the Lagrangian function

$$L(x, u, v) = f(x) - u^T g(x) - v^T h(x)$$

where $u \geq 0$.

Lagrangian relaxation: 2

- A *continuous two person zero sum game* between two players 1 and 2. X and $Y = \{(u, v) : u \geq 0\}$ are the *strategy sets* for players 1 and 2. $L(x, u, v)$ is the *payoff*, i.e., the amount that player 1 gets from 2.
- Player 1 tries to maximize her minimum payoff. Her problem is

$$\max_{x \in X} \min_{(u, v) \in Y} L(x, u, v)$$

which is the original optimization problem (P).

- Player 2 tries to minimize his maximum payoff. His problem is

$$\min_{(u, v) \in Y} \max_{x \in X} L(x, u, v)$$

which is the Lagrangian dual problem.

Properties of Lagrangian dual problem

- The Lagrangian dual problem (D) can also be written as

$$\min_{u \geq 0, v} \theta(u, v)$$

where $\theta(u, v) = \max\{f(x) - u^T g(x) - v^T h(x) : x \in X\}$.

- The objective value of any feasible solution (u, v) to (D) provides an upper bound on the optimal objective $f(x^*)$ of (P). The optimal objective $\theta(u^*, v^*)$ of (D) provides the *best upper bound*.
- $\theta(u^*, v^*) = f(x^*)$ for convex problems with constraint qualification.
- For nonconvex problems, $\theta(u^*, v^*) > f(x^*)$ (duality gap).
- Note that $\theta(u, v)$ is the pointwise supremum of a set of affine functions in u and v . (D) is *always convex* (regardless of whether (P) is convex or not!) but it is in general a *nonsmooth* problem.

Example 1: Getting good upper bounds

- Consider the following integer quadratic program

$$\begin{aligned} \max \quad & x^T Q x \\ \text{s.t.} \quad & x_i^2 = 1, \quad i = 1, \dots, n. \end{aligned}$$

- We compute the functions

$$\begin{aligned} L(x, v) &= x^T Q x - \sum_{i=1}^n v_i (x_i^2 - 1) \\ &= x^T (Q - \text{Diag}(v)) x + e^T v \\ \theta(v) &= e^T v + \max_x x^T (Q - \text{Diag}(v)) x \\ &= e^T v \text{ if } (\text{Diag}(v) - Q) \succeq 0. \end{aligned}$$

Note, that $\theta(v) = \infty$ otherwise.

- The Lagrangian dual problem is

$$\begin{aligned} \min \quad & e^T v \\ \text{s.t.} \quad & (\text{Diag}(v) - Q) \succeq 0. \end{aligned}$$

Example 1: Getting good upper bounds

- This gives the pair of SDPs

$$\begin{array}{ll} \min & \sum_i v_i \\ \text{s.t.} & S = \text{Diag}(v) - Q \\ & S \succeq 0 \end{array}$$

$$\begin{array}{ll} \max & Q \bullet X \\ \text{s.t.} & X_{ii} = 1, \quad i = 1, \dots, n \\ & X \succeq 0 \end{array}$$

which provides the desired upper bound. SDPs can be solved efficiently using interior-point methods.

- A feasible x in the integer program also gives a feasible $X = xx^T$ in the primal SDP, which is a relaxation of the integer program.
- Goemans & Williamson developed a 0.878 approximation algorithm for the maxcut problem which solves the above SDP relaxation.
- The SDP relaxation can be used in a *conic branch-cut-price* algorithm to solve the original integer program to optimality.

Example 2: Exploiting problem structure

- Consider the problem

$$\begin{aligned} \max \quad & \sum_{i=1}^r f_i(x_i) \\ \text{s.t.} \quad & \sum_{i=1}^r h_i(x_i) = 0 \\ & x_i \in X_i \end{aligned}$$

where $f(x)$ and $g(x)$ are *block separable* and X is *decomposable*, i.e.,
 $X = \{x = (x_1, \dots, x_r) : x_i \in X_i\}$

- The objective function in the Lagrangian dual problem is

$$\begin{aligned} \theta(v) &= \max_{x_i \in X_i} \sum_{i=1}^r (f_i(x_i) - v^T h_i(x_i)) \\ &= \sum_{i=1}^r \max_{x_i \in X_i} (f_i(x_i) - v^T h_i(x_i)) \end{aligned}$$

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Example 2: Exploiting problem structure

- We have $\theta(v) = \sum_{i=1}^r \theta_i(v)$ where

$$\theta_i(v) = \max_{x_i \in X_i} (f_i(x_i) - v^T h_i(x_i)).$$

Thus, computing $\theta(v)$ can be done in parallel.

- Thus, the Lagrangian dual problem

$$\min_v \theta(v)$$

can be solved more quickly than the original problem.

- Problems with this structure occur routinely; examples include unit commitment problem, multicommodity flows, two stage stochastic programs with recourse, etc.

Solving the Lagrangian dual problem

- The dual problem is

$$\min_{u \geq 0, v} \theta(u, v)$$

where $\theta(u, v) = \max\{f(x) - u^T g(x) - v^T h(x) : x \in X\}$.

- The inner maximization problem (*subproblem*) can be solved using a general purpose solver (maximizing $f(x)$ over X is easy!). A solution also gives the function value and a subgradient of $\theta(u, v)$ at a given point (u, v) .
- The outer problem (*master problem*) is a nonsmooth problem. Techniques to solve it include: *subgradient*, *cutting plane/column generation*, *bundle/trust region*, and *augmented Lagrangian* methods (see Kartik's NA seminar talk on October 10th!).
- A solution to the primal problem (P) can also be recovered from the solution to the dual problem.

Connections to my current research

Two ongoing research projects

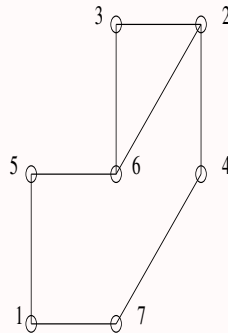
- Developing a conic interior point decomposition approach for structured block angular SDPs in a parallel computing environment
(Example 2 in Lagrangian relaxation).
- Developing semidefinite programming based branch-cut-price algorithms for solving mixed integer and nonconvex problems to optimality
(Example 1 in Lagrangian relaxation).

Preprocessing sparse SDPs into block-angular SDPs: 1

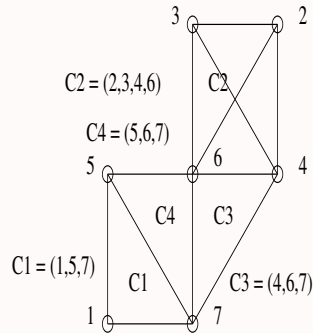
Consider the SDP

$$\begin{aligned} \min \quad & Q \bullet X \\ \text{s.t.} \quad & X_{ii} = 1, \quad i = 1, \dots, n, \\ & X \succeq 0, \end{aligned}$$

where Q is the adjacency matrix of the graph



GRAPH



CHORDAL EXTENSION OF GRAPH

Preprocessing sparse SDPs into block-angular SDPs: 2

Using **matrix completion**, one can reformulate the earlier SDP as

$$\min \sum_{k=1}^4 (Q^k \bullet X^k)$$

$$\text{s.t.} \quad X_{23}^1 - X_{13}^4 = 0,$$

$$X_{34}^2 - X_{12}^3 = 0,$$

$$X_{23}^3 - X_{23}^4 = 0,$$

$$X_{ii}^k = 1, \quad i = 1, \dots, |C_k|, \quad k = 1, \dots, 4,$$

$$X^k \succeq 0, \quad k = 1, \dots, 4,$$

which is in **block-angular** form.

Conclusions

- Lagrangian relaxation technique is a very general technique (i) to obtain upper bounds (ii) exploit problem structure for difficult and large scale optimization problems.
- Both facets form important components of my research.
- Important algorithms in science and engineering solve the Lagrangian dual, examples include *support vector machines* in data mining, *water-filling algorithm* in information theory etc.
- If you are interested, you must definitely drop by my office (HA 235) for a chat! Also, please attend my NA seminar on the 10th of October :)
- Background in numerical linear algebra, parallel and distributed computing, mathematical analysis; and linear, nonlinear, and integer programming will be useful.

Thank you for your attention!
Questions, Comments, Suggestions ?

The slides from this talk are available online at
[http://www4.ncsu.edu/~kksivara/
publications/siam-mgsa-brownbag.pdf](http://www4.ncsu.edu/~kksivara/publications/siam-mgsa-brownbag.pdf)

Bibliography

1. C. LEMARÉCHAL, *Lagrangian relaxation*, In Computational Combinatorial Optimization, M. Jünger and D. Naddef (editors), Springer-Verlag, Heidelberg 2001.
2. R.T. ROCKAFELLAR, *Lagrange multipliers and optimality*, SIAM Review, 35(1993), pp. 183-238.
3. C. LEMARÉCHAL AND F. OUSTRY, *Semidefinite relaxations and lagrangian duality with application to combinatorial optimization*, Rapport de Recherche 3710, INRIA, 1999.
<http://www.inria.fr/rrrt/rr-3710.html>
4. A. RUSZCZYŃSKI, *Nonlinear Optimization*, Princeton University Press, 2006.
5. J.B. HIRIART-URRUTY AND C. LEMARÉCHAL, *Convex Analysis and Minimization Algorithms*, Springer Verlag, New York, 1993.
6. G. NEMHAUSER AND L. WOLSEY, *Integer and Combinatorial Optimization*, Wiley Interscience, 1988.
7. G. DANTZIG AND M. THAPA, *Linear Programming 2: Theory and Extensions*, Springer, New York, 1997.
8. S. BOYD AND L. VANDENBERGHE, *Convex Optimization*, Cambridge University Press 2004.
<http://www.stanford.edu/~boyd/cvxbook/>