

SOLVING POLYNOMIAL OPTIMIZATION PROBLEMS USING REAL ALGEBRAIC GEOMETRY AND SEMIDEFINITE PROGRAMMING

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Polynomial preliminaries 1

1. $p(x) = \sum_{\alpha \in Z_+^n} p_\alpha x_1^{\alpha_1} \dots x_n^{\alpha_n} = \sum_{\alpha \in Z_+^n} p_\alpha x^\alpha$ is a **multivariate polynomial**.
2. The **degree** of the monomial $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ is $|\alpha| = \sum_{i=1}^n \alpha_i$.
3. The **degree** of the polynomial $p(x)$ is the maximum degree of a monomial x^α for which $p_\alpha \neq 0$.
4. **Example:** $x_1^2 x_2^3 x_3 - x_2^4 + 2x_1 x_3 - 1$ is a polynomial in 3 variables with degree 6.
5. Let $S_n^d := \{\alpha \in Z_+^n : |\alpha| \leq d\}$, $|S_n^d| = \binom{n+d}{d}$.
6. One can identify a polynomial $p(x)$ of degree d with its sequence of coefficients $p = (p_\alpha)_{\alpha \in S_n^d}$.
7. The set $\mathbb{R}_d[x_1, \dots, x_n]$ of polynomials of degree $\leq d$ with real coefficients is isomorphic to $\mathbb{R}^{\binom{n+d}{d}}$.

Polynomial preliminaries 2

1. A polynomial $p(x)$ is **nonnegative** if $p(x) \geq 0 \quad \forall x \in \mathbb{R}^n$.
Let $\mathcal{P}_{n,d} = \{p_\alpha : p(x) = \sum_\alpha p_\alpha x^\alpha \geq 0 \quad \forall x \in \mathbb{R}^n, \deg(p(x)) \leq d\}$.
2. A polynomial $p(x)$ is a **sum of squares (SOS)** if

$$p(x) = \sum_i p_i(x)^2.$$

for some polynomials p_i . It is clear that $p(x)$ has even degree $2d$, and each $p_i(x)$ has degree $\leq d$. Let

$\Sigma_{n,d} = \{p_\alpha : p(x) = \sum_\alpha p_\alpha x^\alpha \text{ has an SOS, } \deg(p(x)) \leq d\}$.

3. **Example:** $x_1^2 + x_2^2 + 2x_1x_2 + x_3^6 = (x_1 + x_2)^2 + (x_3^3)^2$ is an SOS.
4. The sets $\mathcal{P}_{n,d}$ and $\Sigma_{n,d}$ are both **closed convex cones**.

When does a nonnegative polynomial have an SOS decomposition?

1. An SOS polynomial is nonnegative but not every nonnegative polynomial is an SOS.
2. Hilbert in 1888 showed that $\mathcal{P}_{n,d} = \Sigma_{n,d}$, i.e., every nonnegative polynomial of degree d in n variables is an SOS when
 - $n = 1$ ($d \geq 2$ and even)
 - $d = 2$ (all n)
 - $n = 2, d = 4$

In all other cases, $\Sigma_{n,d} \subset \mathcal{P}_{n,d}$ (see Chapter 6 in *Squares* by Rajwade).

3. The Motzkin polynomial $x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 + 1$ is nonnegative but *not* an SOS.

Polynomial optimization

1. Let p_0, \dots, p_k be polynomials with real coefficients defined on \mathbb{R}^n . The polynomial optimization problem is

$$\min_{x \in S} p_0(x)$$

where S is a *compact semi-algebraic* set given by

$$S := \{x \in \mathbb{R}^n : p_i(x) \geq 0, i = 1, \dots, k\}.$$

The set S can be nonconvex and even disconnected.

2. Applications in global and combinatorial optimization, statistics, geometry, control, etc.
3. The general problem is *NP-hard*, i.e., intractable.
4. One can rewrite the optimization problem as

$$\max\{t : p_0(x) - t \geq 0 \forall x \in S\}$$

The maxcut problem

- **Maxcut problem:** Given a graph $G = (V, E)$ with nonnegative edge weights $W = (w_{ij})$, find a bipartition $(S, V \setminus S)$ of V , which defines a cut

$$\delta(S) = \{(i, j) : i \in S, j \in V \setminus S\}$$

with maximum weight.

- An integer formulation is

$$\begin{aligned} \max \quad & \frac{1}{4} \sum_{i=1}^n \sum_{j=1}^n w_{ij} (1 - x_i x_j) \\ \text{s.t.} \quad & x_i \in \{-1, 1\}, \quad i = 1, \dots, n, \\ & \max \quad \frac{1}{4} x^T L x \\ & \text{s.t.} \quad x_i^2 = 1, \quad i = 1, \dots, n, \end{aligned}$$

where L is the Laplacian matrix of the graph.

- This is a polynomial optimization problem; we can rewrite the equality constraints as $1 - x_i^2 \geq 0$ and $x_i^2 - 1 \geq 0$.

Underlying paradigm in today's approach

1. Testing whether a polynomial $p(x)$ is *nonnegative* is hard but one can test whether p is an *SOS* efficiently via *semidefinite programming*, although both are convex problems!
CAVEAT: *Convex problems can be hard to solve too!*
2. We want to *approximate the polynomial optimization problem by a hierarchy of semidefinite programs* (Shor (1987), Nesterov, Lasserre, Parrilo (2000)).
3. The semidefinite programs are constructed by *representations of nonnegative polynomials as sums of squares of polynomials*.
4. Semidefinite programs can be *efficiently* solved in polynomial time using interior point methods (Nesterov and Nemirovskii, Alizadeh, Renegar (1990-2000)).
5. The solution of polynomial optimization problems lies at the confluence of *pure* (real algebraic geometry) and *applied* (interior point methods, optimization, and numerical linear algebra) mathematics.

Semidefinite programming

$$\begin{array}{ll} \text{(SDP)} & \min C \bullet X \\ & \text{s.t. } \mathcal{A}(X) = b \\ & X \succeq 0 \end{array}$$

$$\begin{array}{ll} \text{(SDD)} & \max b^T y \\ & \text{s.t. } \mathcal{A}^T y + S = C \\ & S \succeq 0 \end{array}$$

• Notation

- $X, S, C \in \mathcal{S}^n, b \in \mathbb{R}^m$
- $A \bullet B = \text{trace}(AB) = \sum_{i,j=1}^n A_{ij}B_{ij}$ (Frobenius inner product)
- The operator $\mathcal{A} : \mathcal{S}^n \rightarrow \mathbb{R}^m$ and its adjoint $\mathcal{A}^T : \mathbb{R}^m \rightarrow \mathcal{S}^n$ are

$$\mathcal{A}(X) = \begin{pmatrix} A_1 \bullet X \\ \vdots \\ A_m \bullet X \end{pmatrix}, \quad \mathcal{A}^T y = \sum_{i=1}^m y_i A_i$$

where $A_i \in \mathcal{S}^n, i = 1, \dots, m$

Connection between SOS and SDP

Given $p(x) = \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| \leq 2d} p_\alpha x^\alpha$ of degree $2d$. The following are equivalent

1. $p(x)$ has an SOS representation
2. $p(x) = z^T M z$ for some matrix $M = (M_{\beta, \gamma})_{|\beta|, |\gamma| \leq d} \succeq 0$ where $z = (x^\beta)_{|\beta| \leq d}$.
3. The following semidefinite feasibility problem

$$\begin{aligned} M &\succeq 0, \\ \sum_{i, j \in S_n^d: i+j=\alpha} M_{i, j} &= p_\alpha, \quad |\alpha| \leq 2d \end{aligned}$$

is feasible.

This problem is in the primal (SDP) form with matrix size $\binom{n+d}{d}$ and $\binom{n+2d}{2d}$ equality constraints.

Example

Does $p(x) = 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4$ have an SOS?

1. Write $p(x) = z^T M z$ where $z = [1 \ x_1 \ x_2 \ x_1^2 \ x_1x_2 \ x_2^2]$ with rows and columns of M indexed by $\{\{0, 0\}, \{1, 0\}, \{0, 1\}, \{2, 0\}, \{1, 1\}, \{0, 2\}\}$.
2. Equating coefficients gives $M_{11} = 0, M_{12} = 0, M_{13} = 0, 2M_{14} + M_{22} = 0, M_{15} + M_{23} = 0, M_{33} + 2M_{16} = 0, M_{24} = 0, M_{25} + M_{34} = 0, M_{26} + M_{35} = 0, M_{36} = 0, M_{44} = 2, M_{45} = 1, M_{55} + 2M_{46} = -1, M_{56} = 0,$ and $M_{66} = 5$.
3. The SDP feasibility problem with these 15 constraints and $M \succeq 0$ has a feasible solution. So $p(x)$ has an SOS. Indeed,

$$\begin{aligned} p(x) = & 1.0036(0.6070x_1^2 - 0.7736x_1x_2 + 0.1820x_2^2)^2 + \\ & 1.9658(0.7206x_1^2 + 0.6323x_1x_2 + 0.2845x_2^2)^2 + \\ & 5.4266(-0.3351x_1^2 - 0.0416x_1x_2 + 0.9412x_2^2)^2 \end{aligned}$$

This SOS decomposition can be constructed from the eigenvalue and eigenvector information of M .

Unconstrained polynomial optimization

1. Given a polynomial $p(x)$ in n variables and even degree $2d$.

$$\begin{aligned} p_{\min} &= \inf_{x \in \mathbb{R}^n} p(x) \\ &= \sup\{t : p(x) - t \geq 0 \ \forall x \in \mathbb{R}^n\}. \end{aligned}$$

The constraint requires $p(x) - t \in \mathcal{P}_{n,2d}$.

2. A constrained version of this problem is

$$p_{\text{sos}} = \sup\{t : p(x) - t \text{ is SOS}\}.$$

Since $p(x) - t \in \Sigma_{n,2d} \subseteq \mathcal{P}_{n,2d}$, we have $p_{\min} \geq p_{\text{sos}}$.

3. The second problem is the following SDP

$$\begin{aligned} &\sup && t \\ \text{s.t.} &&& M_{0,0} + t = p_0, \\ &&& \sum_{i,j \in S_n^d: i+j=\alpha} M_{i,j} = p_\alpha \ \forall \alpha, \\ &&& M \succeq 0. \end{aligned}$$

Hilbert's 17th problem and general unconstrained case

1. Hilbert's conjecture in 1900:

Does every nonnegative polynomial $p(x)$ have a decomposition as a sum of squares of rational functions?, i.e., does there exist a polynomial $q(x)$ with an SOS decomposition for which $q(x)p(x)$ has an SOS representation.

2. Answer is yes and Emil Artin proved the conjecture in 1927.

3. One can use Hilbert's 17th problem to generate stronger lower bounds in the unconstrained case, by solving

$$p_{sos} = \sup\{t : q(x)(p(x) - t) \text{ and } q(x) \text{ are SOS}\}.$$

with varying degrees for $q(x)$. This can be formulated as an SDP.

The Positivstellensatz 1

1. Consider the following feasibility problem for semialgebraic sets.

Does there exist $x \in \mathbb{R}^n$ such that

$$\begin{aligned} f_i(x) &\geq 0 \quad i = 1, \dots, m, \\ h_j(x) &= 0 \quad j = 1, \dots, p. \end{aligned}$$

2. Consider the dual problem

Do there exist $t_i \in \mathbb{R}[x_1, \dots, x_n]$ and $s_0, s_i, r_{ij}, \dots \in \Sigma$ such that

$$-1 = \sum_i h_i t_i + s_0 + \sum_i s_i f_i + \sum_{i \neq j} r_{ij} f_i f_j + \dots$$

3. The Positivstellensatz is a theorem of alternative for these two systems. (Stengle (1974); see Theorem 4.4.2 in Bochnak, Coste, Roy (1998)).
4. There are no *a priori* guarantees on the degrees of the polynomials t_i, s_0, s_i, r_{ij} etc.

The Positivstellensatz 2

1. Classical theorems of the alternative in optimization such as **Farkas lemma** and **S procedure** (see Boyd and Vandenberghe (2004)) can be derived via the Positivstellensatz (see Strumfels (2002) and Lall (2004-2005)).
2. The dual problem is always *convex* regardless of any convexity assumptions on the original polynomial program.
3. If we consider *bounded* degree polynomials $F = \sum_i h_i t_i$, $G = \sum_i s_i f_i$,

$H = \sum_{i \neq j} r_{ij} f_i f_j$, the dual problem can be posed as a polynomial size semidefinite program. By varying the degree we can construct a hierarchy of semidefinite approximations to the polynomial program.

Constrained multivariate case 1

- Find the best lower bound to

$$\begin{aligned} \min \quad & x^T Q x \\ \text{s.t.} \quad & x_i^2 = 1 \quad i = 1, \dots, n \end{aligned}$$

- This problem can be posed as

$$\max \{t : x^T Q x - t > 0, \quad x_i^2 = 1, \quad i = 1, \dots, n\}.$$

- The constraint set for this problem is nonempty if and only if the set $S = \{(x, t) \in \mathbb{R}^{n+1} : t - x^T Q x \geq 0, \quad x_i^2 - 1 = 0, \quad i = 1, \dots, n\}$ is empty.
- The **Positivstellensatz** with $s_0 = x^T S x$ ($S \succeq 0$), $s_1 = 1$, and $t_i \in \mathbb{R}$ suggests that $S = \emptyset$ if and only if the following system

$$-1 = x^T S x + t - x^T Q x + \sum_i t_i (x_i^2 - 1)$$

with $S \succeq 0$ has a solution.

Constrained multivariate case 2

- This gives the pair of semidefinite programs

$$\begin{array}{ll} \max & \sum_i t_i \\ \text{s.t.} & S = Q - \text{Diag}(t) \\ & S \succeq 0 \end{array}$$

$$\begin{array}{ll} \min & Q \bullet X \\ \text{s.t.} & X_{ii} = 1, \quad i = 1, \dots, n \\ & X \succeq 0 \end{array}$$

which provides the desired lower bound. Note that $X = xx^T$ is feasible in the primal SDP indicating that is a relaxation of the original integer program.

- Stronger lower bounds can be obtained by taking higher degree expressions for s_0 , s_i , and t_i .

Constrained multivariate case 3

1. Consider

$$p_{\min} = \sup\{t : p_0(x) - t \geq 0, \forall x \in S\}$$

where $S = \{x \in \mathbb{R}^n : p_i(x) \geq 0, i = 1, \dots, k\}$.

2. Replace the nonnegativity condition by a stronger condition in terms of SOS, and formulate a hierarchy of lower bounds p_t where

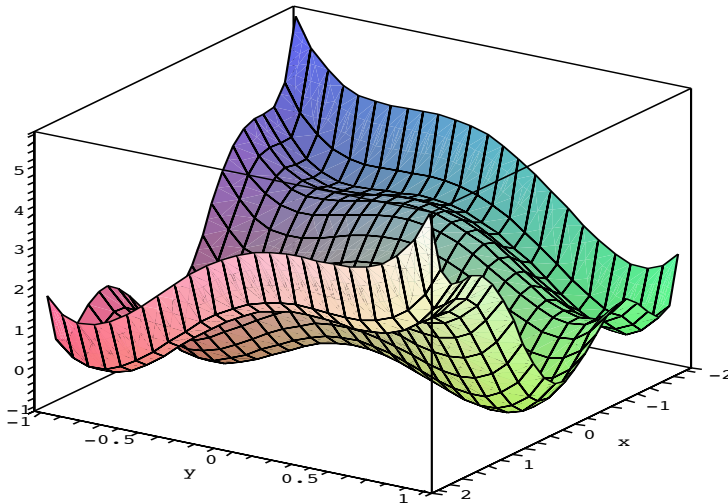
$$p_t = \sup\{t : p_0(x) - t = s_0(x) + \sum_{i=1}^k s_i p_i(x)$$
$$s_0, s_1, \dots, s_k \text{ have an SOS, } \deg(s_0), \deg(s_i p_i) \leq 2t\}$$

$\forall t$, such that $2t \geq \deg(p_0), \deg(p_i)$. Each problem in the hierarchy can be formulated as an SDP. The SOS conditions can be obtained via the Positivstellensatz.

3. We have $p_t \leq p_{t+1} \leq p_{\min}$ with $\lim_{t \rightarrow \infty} p_t = p_{\min}$ if S is compact and satisfies other nice conditions (Lasserre 2000). Finite termination with $\{0, 1\}$ constraints $x_i^2 - x_i = 0$ in the problem (Lasserre 2001).

Demo: Six hump camel back function

Consider minimizing $p(x, y) = x^2(4 - 2.1x^2 + \frac{1}{3}x^4) + xy + y^2(-4 + 4y^2)$.



The function has six local minima, and two global minima at $(0.089, -0.717)$ and $(-0.0898, 0.717)$.

Solving the first SDP in the hierarchy gives both global optimal solutions. (I checked this with Gloptipoly which also has a solution extraction scheme).

Available software for polynomial optimization

- **Gloptipoly**: Global optimization over polynomials with Matlab and **SeDuMi** by Didier Henrion and Jean Lasserre. Available at <http://www.laas.fr/~henrion/software/gloptipoly/>
- **SOSTOOLS** - A sum of squares optimization toolbox for Matlab by Pablo Parrilo et al. Available at <http://www.mit.edu/~parrilo/sostools/>
- Both software packages use the MATLAB environment and the SDP solver **SeDuMi** by Jos Sturm. SeDuMi is available at <http://fewcal.kub.nl/sturm/software/sedumi.html>

Conclusions and future developments

1. The SDP approach to polynomial optimization is competitive with state-of-the-art global optimization software.
2. The approach has been used to devise polynomial time approximation schemes (PTAS) for polynomial optimization problems over the simplex (DeKlerk-Laurent-Parrilo).
3. We also have certificates of global optimality in the approach (see Lasserre). There is also a mechanism to extract the solutions from the dual SDP. See the [Gloptipoly](#) toolbox by Henrion and Lasserre.
4. Solution of large scale SDPs is still a bottleneck! Kartik is currently exploring decomposition and nonsmooth techniques in a high performance computing environment to solve large scale SDPs.
5. Exploiting symmetry to reduce the size of the SDP (Parrilo-Gatermann, DeKlerk-Pasechnik-Schrijver).
6. Exploiting the sparsity of the polynomials to reduce the size of the SDP (Kim-Kojima-Muramatsu-Waki).

**Thank you for your attention!.
Questions, Comments, Suggestions ?**

**The slides from this talk are available online at
[http://www4.ncsu.edu/~kksivara/
publications/ncsu-symb.pdf](http://www4.ncsu.edu/~kksivara/publications/ncsu-symb.pdf)**

Bibliography 1

1. **P.A. PARRILO**, *Semidefinite programming relaxations for semialgebraic problems*, Mathematical Programming, 96(2003), pp. 293-320.
2. **C. CARAMANIS**, *Non-convex optimization via real algebraic geometry*, http://web.mit.edu/%7Ecmccaram/www/pubs/nonconvex_opt_review.pdf
3. **M. LAURENT**, *Semidefinite programming in polynomial optimization*, Plenary talk in the 2005 SIAM Optimization Conference, Stockholm, May 2005. Slides available at <http://homepages.cwi.nl/~monique/files/siopt2005.ps>
4. **J.B. LASSERRE**, *Global optimization with polynomials and the problem of moments*, SIAM Journal on Optimization, 11(2001), pp. 796-817.
5. **J.B. LASSERRE**, *An explicit equivalent positive semidefinite program for nonlinear 0-1 programs*, SIAM Journal on Optimization, 12(2002), pp. 756-769.
6. **A. RAJWADE**, *Squares*, Volume 171 of London Math Society Lecture Note Series, Cambridge University Press, 1993.
7. **J. BOCHNAK**, **M. COSTE**, **M-F. ROY**, *Real Algebraic Geometry*, Springer-Verlag Berlin Heidelberg 1998.
8. **P.A. PARRILO AND B. STRUMFELS**, *Minimizing polynomial functions*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Volume 60, 2003, pp. 83-99.

Bibliography 2

9. **B. STRUMFELS**, *Solving systems of polynomial equations*, CBMS regional conference series in mathematics, no 97, American Mathematical Society, 2002.
10. **P. A. PARRILO**, *Sums of squares programs and polynomial inequalities*, SIAG/OPT Views-and-News, Volume 15 Number 2, October 2004. Available at http://www.mac.uc.pt/siagopt/vn15_2.pdf
11. **D. HENRION AND J.B. LASSERRE**, *Gloptipoly: Global optimization over polynomials with Matlab and SeDuMi*, <http://www.laas.fr/~henrion/papers/gloptipoly.pdf>
12. **S. PRAJNA, A. PAPACHRISTODOULOU, P. SEILER, AND P. PARRILO**, *SOSTOOLS: Sum of squares optimization toolbox for MATLAB*, <http://www.mit.edu/~parrilo/sostools/sostools.pdf>
13. **S. LALL**, *Engr 210b Advanced Topics in Computation for Control*, Autumn quarter, 04-05, Stanford University. Lecture notes available at <http://tableau.stanford.edu/~lall/courses/current/engr210b/index.html>
14. **B. REZNICK**, *Some concrete aspects of Hilbert's 17th problem*, vol. 253, American Mathematical Society, 2000, pp. 251-272.
15. **S. BOYD AND L. VANDENBERGHE**, *Convex Optimization*, Cambridge University Press 2004.
16. **Y. NESTEROV**, *Squared functional systems and optimization problems*, High Performance Optimization, Kluwer Academic Publishers, 2000, pp. 405-440.

Bibliography 3

17. E. DEKLERK, M. LAURENT, AND P.A. PARRILO, *A PTAS for the minimization of polynomials of fixed degree over the simplex*, Available at <http://homepages.cwi.nl/~monique/files/ptasrev.ps>
18. E. DEKLERK, D.V. PASECHNIK, AND A. SCHRIJVER, *Reduction of symmetric semidefinite programs using the regular *-representation*, Available at http://www.optimization-online.org/DB_HTML/2005/03/1083.html
19. K. GATERMANN AND P.A. PARRILO, *Symmetry groups, semidefinite programs and sums of squares*, *Journal of Pure and Applied Algebra*, 192(2004), pp. 95-128.
20. H. WAKI, S. KIM, M. KOJIMA, M. MURAMATSU, *Sums of squares and semidefinite programming relaxations for polynomial problems with structured sparsity*, Available at http://www.optimization-online.org/DB_HTML/2004/10/988.html
21. D. HENRION AND J.B. LASSERRE, *Detecting global optimality and extracting solutions in Glop-tiPoly*, Available at <http://www.laas.fr/~henrion/papers/extract.pdf>