

A CONIC INTERIOR POINT DECOMPOSITION APPROACH FOR LARGE SCALE SEMIDEFINITE PROGRAMMING

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Conic programming

$$\begin{array}{ll} \text{(P)} & \min \quad c^T x \\ & \text{s.t.} \quad Ax = b \\ & \quad \quad x \in \mathcal{K} \end{array}$$

$$\begin{array}{ll} \text{(D)} & \max \quad b^T y \\ & \text{s.t.} \quad A^T y + s = c \\ & \quad \quad s \in \mathcal{K} \end{array}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $\mathcal{K} = \mathcal{K}_1 \times \dots \times \mathcal{K}_r$

- $r = 1, \mathcal{K} = \mathbb{R}_+^n = \{x \in \mathbb{R}^n : x \geq 0\}$ **LP**
Very large LPs ($m, n \leq \mathbf{1,000,000}$) solvable by the simplex method and/or IPMs.
- $\mathcal{K}_i = \mathbb{Q}_+^{n_i} = \{x \in \mathbb{R}^{n_i} : x_1 \geq \|x_{2:n_i}\|\}$ **SOCP**
Large SOCPs ($m, n \leq \mathbf{100,000}$) solvable by IPMs.
- $\mathcal{K}_i = \mathcal{S}_+^{n_i} = \{X \in \mathcal{S}^{n_i} : X \succeq 0\}$ **SDP**
Medium sized SDPs ($m, n \leq \mathbf{1000}$) solvable by IPMs.
(Beyond 10,000 seems impossible today!)

Motivation

- (a) Solve **large scale structured semidefinite programs (SDP)** arising in science and engineering. Typically, these SDPs need not be solved very accurately.
- (b) The technique is to **iteratively** solve an SDP between a **mixed conic master problem** over linear, second order, and semidefinite cones; and **distributed subproblems** (smaller SDPs) in a high performance computing environment.
- (c) Improve the **scalability** of interior point methods (IPMs) by applying them instead on the smaller master problem, and subproblems (which are solved in parallel!)

This is our **conic interior point decomposition** scheme.

Semidefinite programs with a decomposable structure

1. **Preprocessed SDPs after matrix completion:**
Fukuda-Kojima-Nakata-Murota (2000), Lu-Nemirovskii-Monteiro (2004)
2. **Stability analysis of interconnected subsystems:**
Langbort-D'Andrea-Xiao-Boyd (2003)
3. **Stochastic semidefinite programs:** Mehrotra-Ozëvin (2005)
4. **Polynomial optimization problems with structured sparsity:**
Waki-Kim-Kojima-Muramatsu (2005)
5. **Semidefinite programs arising in lift and project schemes:**
Rendl-Sotirov (2003) and Burer-Vandenbussche (2006)

Semidefinite programming

$$\begin{array}{ll} \text{(SDP)} & \min C \bullet X \\ & \text{s.t. } \mathcal{A}(X) = b \\ & X \succeq 0 \end{array}$$

$$\begin{array}{ll} \text{(SDD)} & \max b^T y \\ & \text{s.t. } \mathcal{A}^T y + S = C \\ & S \succeq 0 \end{array}$$

• Notation

- $X, S, C \in \mathcal{S}^n, b \in \mathbb{R}^m$
- $A \bullet B = \text{trace}(AB) = \sum_{i,j=1}^n A_{ij}B_{ij}$ (Frobenius inner product)
- The operator $\mathcal{A} : \mathcal{S}^n \rightarrow \mathbb{R}^m$ and its adjoint $\mathcal{A}^T : \mathbb{R}^m \rightarrow \mathcal{S}^n$ are

$$\mathcal{A}(X) = \begin{pmatrix} A_1 \bullet X \\ \vdots \\ A_m \bullet X \end{pmatrix}, \quad \mathcal{A}^T y = \sum_{i=1}^m y_i A_i$$

where $A_i \in \mathcal{S}^n, i = 1, \dots, m$

Assumptions

(1) A_1, \dots, A_m are **linearly independent** in \mathcal{S}^n .

(2) **Slater condition** for (SDP) and (SDD):

$$\{X \in \mathcal{S}^n : \mathcal{A}(X) = b, X \succ 0\} \neq \emptyset$$

$$\{(y, S) \in \mathbb{R}^m \times \mathcal{S}^n : \mathcal{A}^T y + S = C, S \succ 0\} \neq \emptyset$$

(3) One of the constraints in (SDP) is $I \bullet X = 1$ ($\text{trace}(X) = 1$).

(Every SDP with a bounded primal feasible set can be reformulated to satisfy this assumption)

Conic Dantzig-Wolfe decomposition

- Our semidefinite programs are:

$$\begin{array}{ll} \text{(SDP)} & \min C \bullet X \\ & \text{s.t. } \mathcal{A}(X) = b \\ & \text{trace}(X) = 1 \\ & X \succeq 0 \end{array}$$

$$\begin{array}{ll} \text{(SDD)} & \max b^T y + z \\ & \text{s.t. } \mathcal{A}^T y + zI + S = C \\ & S \succeq 0 \end{array}$$

- Consider the set:

$$\mathcal{E} = \{X \in \mathcal{S}^n : \text{trace}(X) = 1, X \succeq 0\}$$

- The extreme points of \mathcal{E} are:

$$\{vv^T : v \in \mathbb{R}^n, v^T v = 1\} \text{ (Infinite number)}$$

Conic Dantzig-Wolfe decomposition:

Any $X \in \mathcal{E}$ can be written as:

(1) $X = \sum_j \lambda_j d_j d_j^T$ where $\sum_j \lambda_j = 1, \lambda_j \geq 0$

$X \succeq 0$ is replaced by $\lambda_j \geq 0, \forall j$ (Semi-infinite LP formulation)

(2) $X = \sum_j P_j V_j P_j^T$ where $\sum_j \text{trace}(V_j) = 1, V_j \in \mathcal{S}_+^2$

$X \succeq 0$ is replaced by $V_j \succeq 0, \forall j$ (Semi-infinite SOCP formulation)

(A 2×2 SDP constraint can be written as a size 3 SOCP constraint)

(3) $X = \sum_j P_j V_j P_j^T$ where $\sum_j \text{trace}(V_j) = 1, V_j \in \mathcal{S}_+^{r_j}, r_j \geq 3$

$X \succeq 0$ is replaced by $V_j \succeq 0, \forall j$ (Semi-infinite SDP formulation)

A 2×2 SDP constraint is an SOCP constraint of size 3

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \succeq 0$$

\Leftrightarrow

$$X_{11} \geq 0, \quad X_{22} \geq 0, \quad X_{11}X_{22} - X_{12}^2 \geq 0$$

\Leftrightarrow

$$\begin{pmatrix} X_{11} + X_{22} \\ 2X_{12} \\ X_{11} - X_{22} \end{pmatrix} \succeq_Q 0$$

Conic Dantzig-Wolfe decomposition: Master problem

The **decomposed** conic problem over LP, SOCP and SDP blocks is:

Primal

$$\begin{aligned}
 \min \quad & \sum_{i=1}^{n_l} c_{li} x_{li} + \sum_{j=1}^{n_q} c_{qj}^T x_{qj} + \sum_{k=1}^{n_s} C_{sk} \bullet X_{sk} \\
 \text{s.t.} \quad & \sum_{i=1}^{n_l} A_{li} x_{li} + \sum_{j=1}^{n_q} A_{qj} x_{qj} + \sum_{k=1}^{n_s} \mathcal{A}_{sk}(X_{sk}) = b \\
 & \sum_{i=1}^{n_l} x_{li} + \sum_{j=1}^{n_q} x_{qj} + \sum_{k=1}^{n_s} \text{trace}(X_{sk}) = 1 \\
 & x_{li} \geq 0, \quad i = 1, \dots, n_l \\
 & x_{qj} \preceq_Q 0, \quad j = 1, \dots, n_q \\
 & X_{sk} \preceq 0, \quad k = 1, \dots, n_s
 \end{aligned}$$

Dual

$$\begin{aligned}
 \max \quad & b^T y + z \\
 \text{s.t.} \quad & A_{li}^T y + z + s_{li} = c_{li}, \quad i = 1, \dots, n_l \\
 & A_{qj}^T y + z \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s_{qj} = c_{qj}, \quad j = 1, \dots, n_q \\
 & \mathcal{A}_{sk}^T y + z I_{r_k} + S_{sk} = C_{sk}, \quad k = 1, \dots, n_s \\
 & s_{li} \geq 0, \quad i = 1, \dots, n_l \\
 & s_{qj} \preceq_Q 0, \quad j = 1, \dots, n_q \\
 & S_{sk} \preceq 0, \quad k = 1, \dots, n_s
 \end{aligned}$$

Conic Dantzig-Wolfe decomposition: Separation Oracle

Input: (y^*, z^*) a feasible point for dual master problem.

- If $\lambda_{\min}(S^*) \geq 0$, report feasibility **STOP**.
- Else, solve the following subproblem for X^* :

$$\begin{array}{ll} \min & (C - \mathcal{A}^T y^* - z^* I) \bullet X \\ \text{s.t.} & I \bullet X = 1 \\ & X \succeq 0 \end{array}$$

Factorize $X^* = DMD^T$ with $M \succ 0$.

Return the cut $D^T(C - \mathcal{A}^T y - zI)D \succeq 0$ with $D \in \mathbb{R}^{n \times r}$.

- If $r = 1$ is an LP cut.
- If $r = 2$ is an SOCP cut.
- If $r \geq 3$ is an SDP cut of small size.

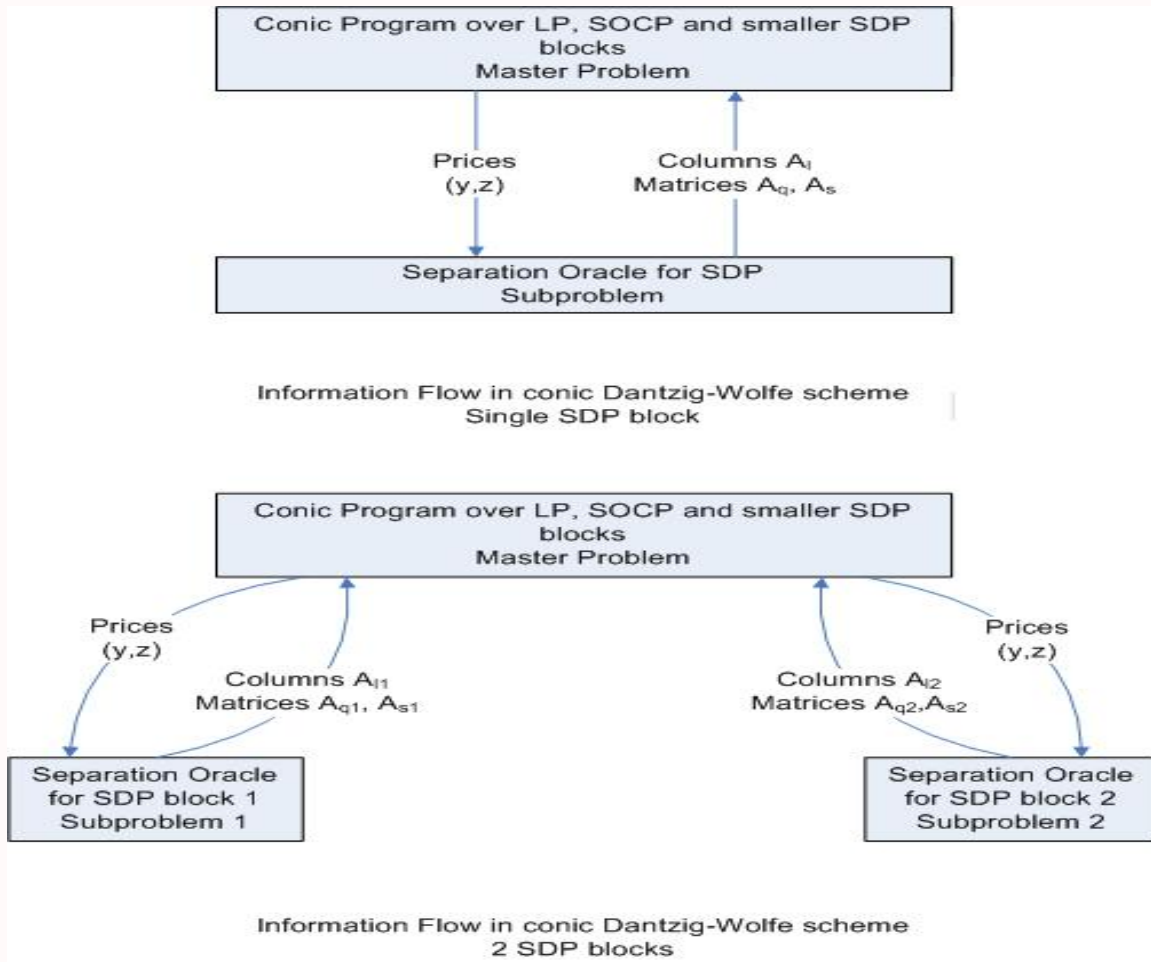


Figure 1: Conic Dantzig-Wolfe Algorithm

Choice of the query point: I

- In the original Dantzig-Wolfe (Kelley) scheme the query point (y, z) is an **optimal solution** to the dual master problem. This scheme has a **poor** rate of convergence.
- Better to solve the master problem approximately initially, and gradually tighten this tolerance as the algorithm proceeds.

– Initially, the master problem is a poor approximation to the SDP problem.

Weak tolerance \Rightarrow Central dual iterates (y, z)
 \Rightarrow Oracle returns better cuts

– As the algorithm proceeds, the master problem approximations get tighter.

Tight tolerance \Rightarrow More emphasis on the objective function
 \Rightarrow Convergence to an optimal solution

Choice of the query point : 2

Our adaptive strategy:

- Solve the master problem to a tolerance TOL for (x^*, y^*, z^*, s^*) . Compute the following parameters:

$$\begin{aligned} \text{GAPTOL}(x^*, s^*) &= \frac{x^{*T} s^*}{\max\{1, \frac{1}{2}(c^T x^* + b^T y^*)\}} \\ \text{INF}(y^*, z^*) &= \frac{\lambda_{\min}(C - A^T y^* - z^* I)}{\max\{1, \frac{1}{2}(c^T x^* + b^T y^*)\}} \\ \text{OPT}(x^*, y^*, z^*, s^*) &= \max\{\text{GAPTOL}(x^*, s^*), \text{INF}(y^*, z^*)\} \end{aligned}$$

- If $\text{OPT}(x^*, y^*, z^*, s^*) < \text{TOL}$, we lower TOL by a constant factor. More precisely, $\text{TOL} = \mu \times \text{OPT}(x^*, y^*, z^*, s^*)$ with $0 < \mu < 1$. Else, TOL remains unchanged.

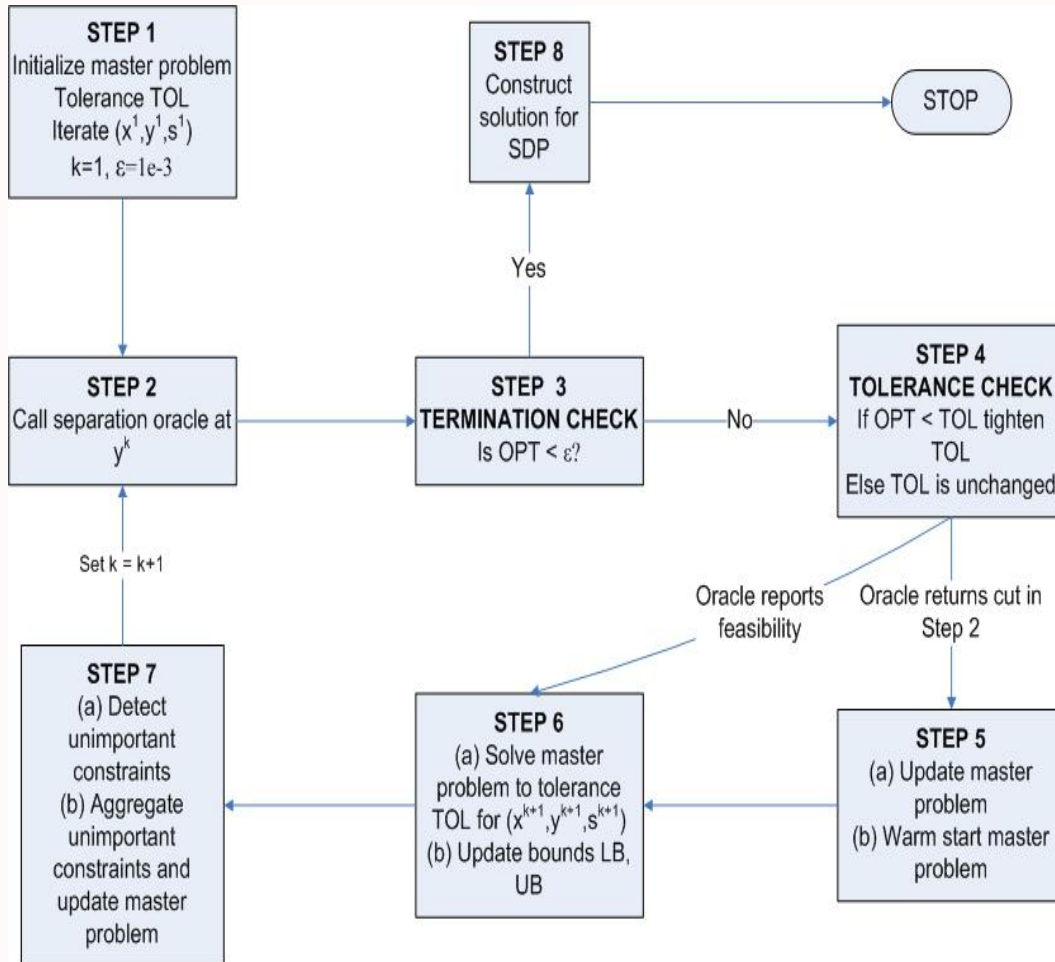
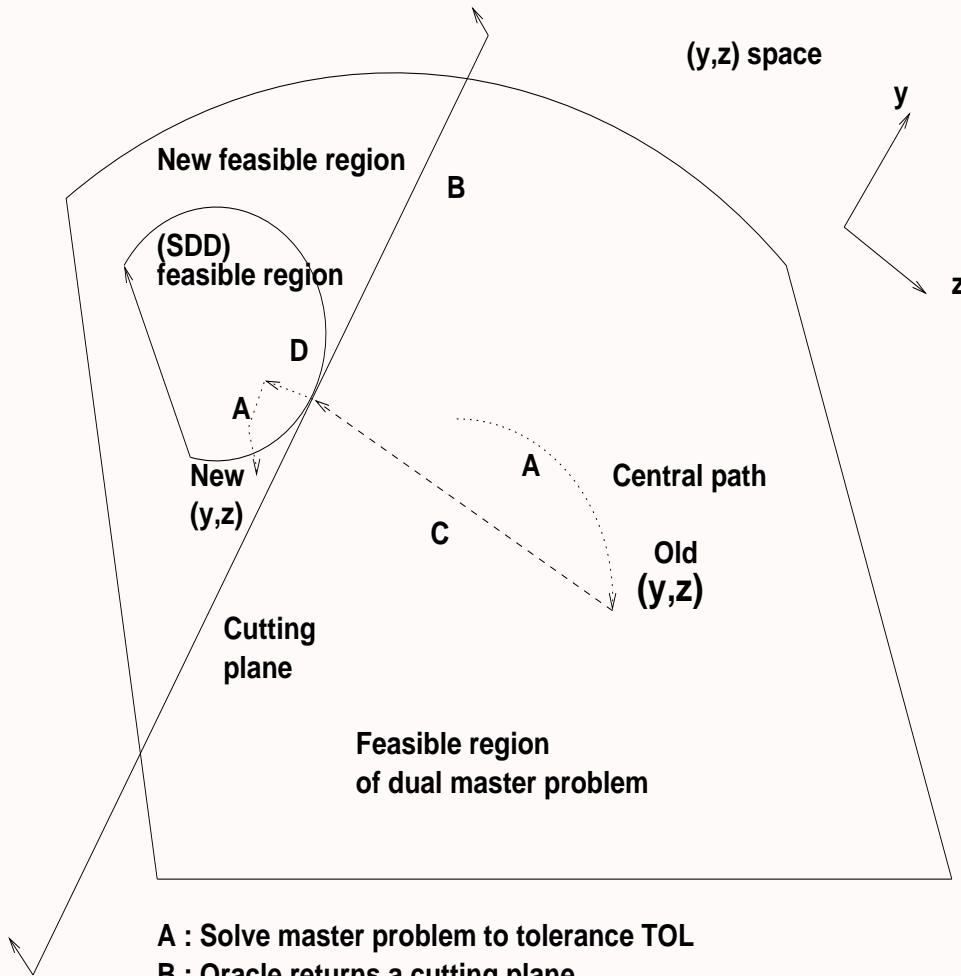


Figure 2: The complete algorithm



- A** : Solve master problem to tolerance TOL
- B** : Oracle returns a cutting plane
- C** : Restoring dual feasibility
- D** : Warm start

Figure 3: One iteration of algorithm

Upper and lower bounds

- The objective value of the primal master problem in every iteration gives an upper bound on the SDP objective value. This is the objective value of the dual master problem plus the duality gap.
- Given a solution (y^*, z^*) to the dual master problem in every iteration, a lower bound is computed as follows:
 - Compute $\lambda^* = \lambda_{\min}(S^*)$, where $S^* = (C - \mathcal{A}^T y^* - z^* I)$.
 - Set $y^{lb} = y^*$ and $z^{lb} = z^* + \lambda^*$.
 - A lower bound on the SDP objective value is then $b^t y^{lb} + z^{lb}$

Note: (y^{lb}, z^{lb}) is a feasible point in (SDD).
- We could also terminate if the difference between these bounds is small.

Warm start after adding a matrix: 1

- The solution to the old master problem is $(x_l^*, x_s^*, y^*, s_l^*, s_s^*)$. Consider adding a semidefinite cut $a_s^T y \preceq d$ in the dual master problem. The new master problem is :

$$\begin{array}{ll} \min & c_l^T x_l + c_s^T x_s + d^T \beta \\ \text{s.t.} & A_l x_l + A_s x_s + a_s \beta = b \\ & x_l \geq 0, \quad \beta \succeq 0 \\ & x_s \succeq 0. \end{array}$$

$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & s_l = c_l - A_l^T y \geq 0 \\ & s_s = c_s - A_s^T y \succeq 0 \\ & \gamma = d - a_s^T y \succeq 0 \end{array}$$

- We have $\gamma^* \not\geq 0$. We can perturb y^* (lower bounding) to generate y^{lb} so that $\gamma^{lb} \succeq 0$.
- The perturbed point $(x_l^*, x_s^*, \beta^*, y^{lb}, s_l^{lb}, s_s^{lb}, \gamma^{lb})$ is feasible in the new master problem with $\beta^* \succeq 0$ and $\gamma^{lb} \succeq 0$, but NOT STRICTLY!
- We want to increase β^* and γ^{lb} , making them strictly feasible, while limiting the variation in the other variables.

Warm start after adding a matrix: 2

- We solve the following problems

$$\begin{aligned} \text{(WSP)} \quad & \max \log \det \beta \\ \text{s.t.} \quad & A_l \Delta x_l + A_s \Delta x_s + a_s \beta = 0 \\ & \sqrt{\|D_l^{-1} \Delta x_l\|^2 + \|D_s^{-1} \Delta x_s\|^2} \leq 1 \\ & \beta \succeq 0. \end{aligned}$$

$$\begin{aligned} \text{(WSD)} \quad & \max \log \det \gamma \\ \text{s.t.} \quad & a_s^T \Delta y + \gamma = 0 \\ & \sqrt{\|D_l A_l^T \Delta y\|^2 + \|D_s A_s^T \Delta y\|^2} \leq 1 \\ & \gamma \succeq 0. \end{aligned}$$

for $(\Delta x_l, \Delta x_s, \beta)$ and $(\Delta y, \gamma)$ respectively. Here D_l and D_s are appropriate primal-dual scaling matrices for LP and SDP blocks at $(x_l^*, x_s^*, s_l^{lb}, s_s^{lb})$.

- Compute $(\Delta s_l, \Delta s_s) = (-A_l^T \Delta y, -A_s^T \Delta y)$.

Warm start after adding a matrix: 3

- Compute

$$\begin{aligned}
 \Delta x_l &= -D_l^2 A_l^T (A_l D_l^2 A_l^T + A_s D_s^2 A_s^T)^{-1} a_s \beta \\
 \Delta x_s &= -D_s^2 A_s^T (A_l D_l^2 A_l^T + A_s D_s^2 A_s^T)^{-1} a_s \beta \\
 \Delta y &= -(A_l D_l^2 A_l^T + A_s D_s^2 A_s^T)^{-1} a_s \beta \\
 \gamma &= V \beta
 \end{aligned}$$

where $\beta \in \mathcal{S}^p$ is the solution to

$$\min \left\{ \frac{p}{2} \beta^T V \beta - \log \det \beta : \beta \succeq 0 \right\}$$

where $V = a_s^T (A_l D_l^2 A_l^T + A_s D_s^2 A_s^T)^{-1} a_s$.

- Finally set

$$\begin{aligned}
 (x_l^{st}, x_s^{st}, \beta^{st}) &= (x_l^* + \kappa \alpha_{max}^p \Delta x_l, x_s^* + \kappa \alpha_{max}^p \Delta x_s, \kappa \alpha_{max}^p \beta) \\
 y^{st} &= y_l^{lb} + \kappa \alpha_{max}^d \Delta y \\
 (s_l^{st}, s_s^{st}, \gamma^{st}) &= (s_l^{lb} + \kappa \alpha_{max}^d \Delta s_l, s_s^{lb} + \kappa \alpha_{max}^d \Delta s_s, \kappa \alpha_{max}^d \gamma)
 \end{aligned}$$

where $\kappa \in (0, 1)$ and $\alpha_{max}^p, \alpha_{max}^d$ are the maximal primal and dual step lengths respectively.

Aggregating unimportant blocks

- Use a primal-dual scaling measure to evaluate blocks:
 - For the i th linear block the measure is $\frac{x_i}{s_i}$.
 - For the j th semidefinite block the measure is $\frac{\text{trace}(x_j)}{\text{trace}(s_j)}$.
- Blocks with small measure are aggregated in an LP block.
- The resulting master problem is

$$\begin{aligned} \min \quad & c_{agg}x_{agg} + c_l^T x_l + c_s^T x_s \\ \text{s.t.} \quad & A_{agg}x_{agg} + A_l x_l + A_s x_s = b \\ & x_l \geq 0, \quad x_{agg} \geq 0 \\ & x_s \succeq 0. \end{aligned}$$

$$\begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & s_l = c_l - A_l^T y \geq 0 \\ & s_s = c_s - A_s^T y \succeq 0 \\ & s_{agg} = c_{agg} - A_{agg}^T y \geq 0 \end{aligned}$$

Computational Results

- Matlab code with the following features:
 - Master problem contains linear and semidefinite blocks only.
 - **SDPT3** is used as conic solver for master problem with H.K.M. scaling
 - Oracle is implemented using MATLAB's Lanczos solver (**eigs**), which computes the $r = \lfloor \frac{\sqrt{m}}{2} \rfloor$ most negative eigenvalues and eigenvectors of S in each iteration.
 - Solve the problem for m iterations, or until $TOL = 1e-3$, whichever comes earlier.
- All computational results on a 2.4 GHz processor with 1 GB of memory.

Computational results for Max-Cut

Problem	n	Opt value	LP cones	SDP cones	Lower bound	ϵ	Time (h:m:s)
mcp100	100	-226.16	101	152(31)	-226.32	1.6e-3	16
mcp124-1	124	-141.99	125	187(48)	-141.87	1.5e-3	34
mcp124-2	124	-269.88	125	173(36)	-270.05	1.4e-3	25
mcp124-3	124	-467.75	125	174(35)	-468.07	1.5e-3	22
mcp124-4	124	-864.41	125	189(41)	-864.99	1.5e-3	21
mcp250-1	250	-317.26	251	339(84)	-317.46	1.3e-3	2:03
mcp250-2	250	-531.93	251	343(51)	-532.35	1.7e-3	1:05
mcp250-3	250	-981.17	251	307(44)	-981.86	1.6e-3	49
mcp250-4	250	-1681.96	251	301(43)	-1683.1	1.6e-3	47
mcp500-1	500	-598.15	501	735(115)	-598.56	1.6e-3	12:29
mcp500-2	500	-1070.06	501	634(70)	-1070.70	1.5e-3	6:18
mcp500-3	500	-1847.97	501	570(57)	-1849.20	1.6e-3	4:33
mcp500-4	500	-3566.73	501	523(49)	-3569.10	1.6e-3	2:51
toruspm3-8-50	512	-527.80	513	681(84)	-528.10	1.4e-3	9:28
torusg3-8	512	-457.36	513	605(89)	-457.60	1.4e-3	13.11
maxG11	800	-629.16	801	734(84)	-629.39	1.4e-3	1:49:55
maxG51	1000	-4003.81	1001	790(54)	-4008.90	1.7e-3	32:24

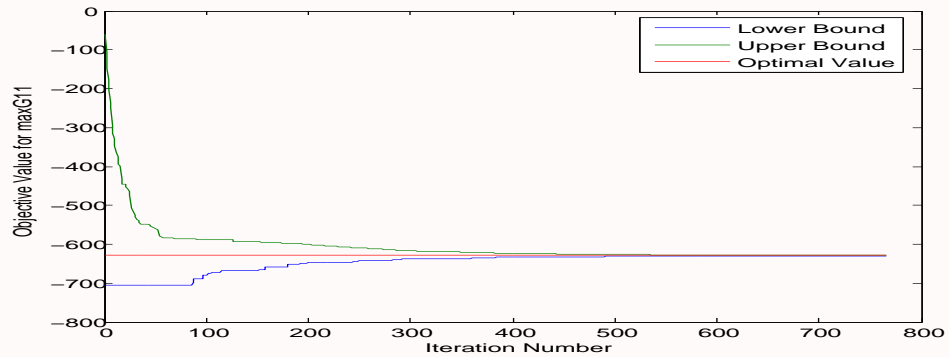


Figure 4: Variation of bounds for maxG11 problem

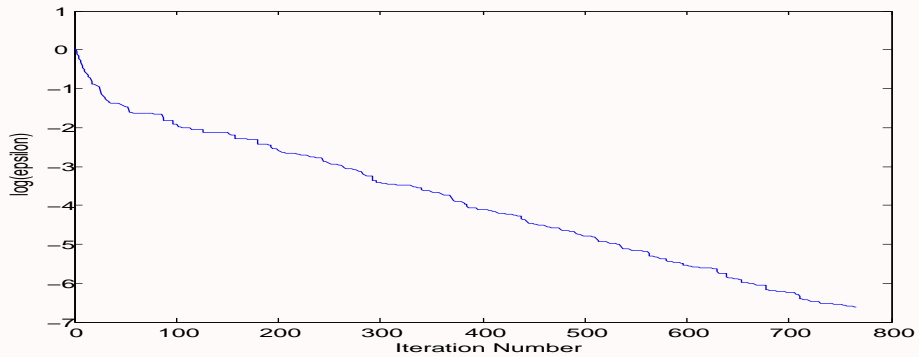


Figure 5: Variation of error for maxG11 problem

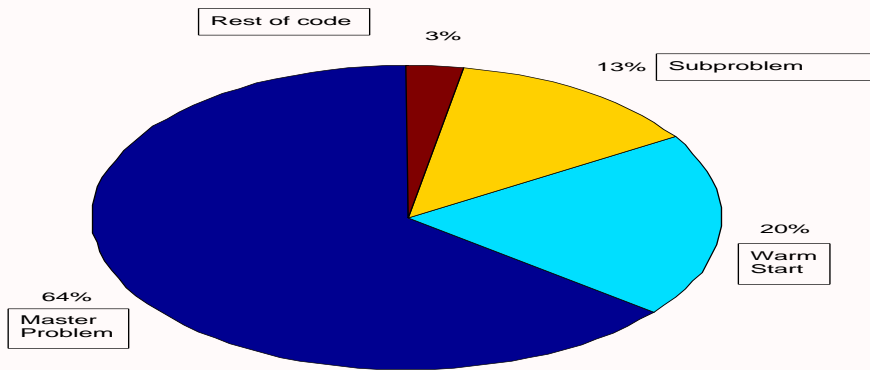


Figure 6: Distribution of times for maxG11 problem

Computational results on random semidefinite programs

$$\begin{aligned}
 & \max \quad b^T y + z \\
 & \text{s.t.} \quad S = (C - \mathcal{A}^T y - zI) \succeq 0 \\
 & \quad \quad l \leq y \leq u.
 \end{aligned}$$

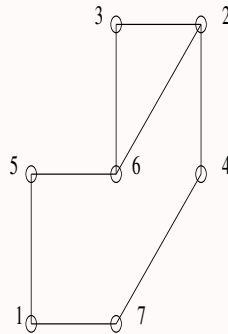
Prob	n	m	Den	LP cones	SDP cones	Our LB	Our (h:m:s)	SDPT3 LB	SDPT3 (h:m:s)
test-1	100	501	0.1	1100	45(9)	424.68	1:06	424.97	43
test-2	100	501	0.9	1100	140(28)	181.91	5:21	182.02	1:32
test-3	500	101	0.1	700	50(25)	33.36	4:33	33.38	4:31
test-4	500	101	0.9	700	116(29)	-37.99	4:49	-37.96	6:35
test-5	1000	51	0.1	1100	24(12)	7.99	18:40	7.99	16:03
test-6	1000	51	0.9	1100	86(19)	-13.33	13:25	-13.32	17:30
test-7	300	300	0.1	900	105(21)	147.51	2:39	147.56	4:13
test-8	300	300	0.9	900	142(18)	22.91	4:52	22.92	8:11

Preprocessing sparse SDPs into block-angular SDPs: 1

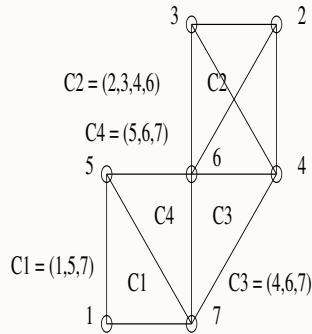
Consider the SDP

$$\begin{aligned} \min \quad & L \bullet X \\ \text{s.t.} \quad & X_{ii} = 1, \quad i = 1, \dots, n, \\ & X \succeq 0, \end{aligned}$$

where L is the adjacency matrix of the graph



GRAPH



CHORDAL EXTENSION OF GRAPH

Preprocessing sparse SDPs into block-angular SDPs: 2

Using **matrix completion**, one can reformulate the earlier SDP as

$$\min \sum_{k=1}^4 (L^k \bullet X^k)$$

$$\text{s.t.} \quad X_{23}^1 - X_{13}^4 = 0,$$

$$X_{34}^2 - X_{12}^3 = 0,$$

$$X_{23}^3 - X_{23}^4 = 0,$$

$$X_{ii}^k = 1, \quad i = 1, \dots, |C_k|, \quad k = 1, \dots, 4,$$

$$X^k \succeq 0, \quad k = 1, \dots, 4,$$

which is in **block-angular** form.

Preliminary results on block angular semidefinite programs

- Results on the IBM Blade Center Linux Cluster (Henry2) at NC State.
- Each of the 175 nodes is a 2.8-3.2 GHz processor with 4 GB of memory.
- Our code is in C and uses MPI for interprocessor communication.
- CPLEX 9.0 was used to solve the master problem and CSDP was used to solve the subproblems.
- 3 digits of accuracy or an upper limit of 2 hours in computations.

Prob	n	n_p	m	Our LB	Our h:m:s (p)	SDPT3 LB	SDPT3 (h:m:s)
mcp124-1	124	60(6)	172	141.92	5(4)	141.99	4
mcp250-1	250	176(9)	381	317.06	3:33(4)	317.26	9
mcp500-1	500	324(19)	1222	597.74	34:39(8)	598.15	36
maxG11	800	216(4)	1208	628.88	8:50(4)	629.16	1:37
maxG32	2000	1022(2)	2820	1566.20	2:0:0(2)	1567.02	1:32:06

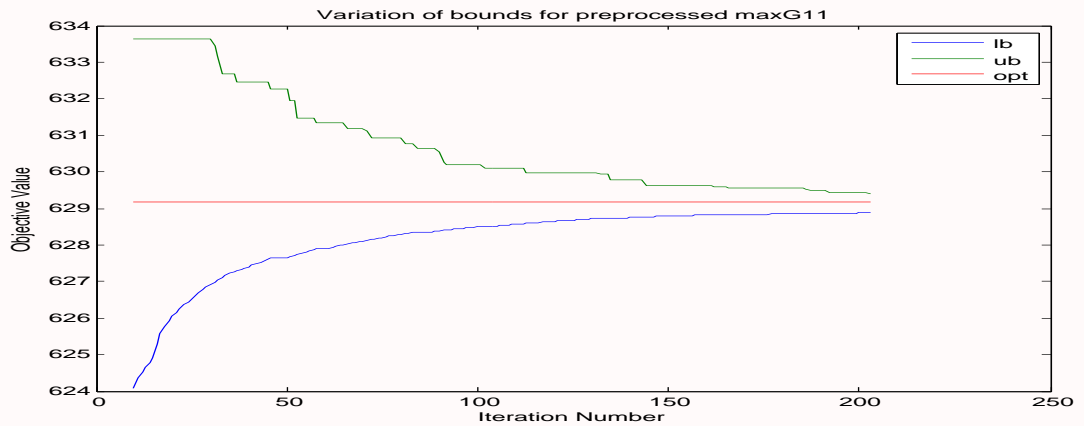


Figure 7: Variation of bounds for preprocessed maxG11 problem

Ongoing and Future work

- Applying the conic interior point decomposition approach for structured block angular SDPs in a parallel computing environment.
- Incorporating the decomposition scheme in the pricing phase of an SDP based conic branch-cut-price scheme for mixed integer and non-convex problems.

Thank you for your attention!

Questions, Comments, Suggestions ?

The slides from this talk are available online at

[http://www4.ncsu.edu/~kksivara/
publications/ismmp2006-kartik.pdf](http://www4.ncsu.edu/~kksivara/publications/ismmp2006-kartik.pdf)

A technical report appears at

[http://www4.ncsu.edu/~kksivara/
publications/conic-ipm-decomposition.pdf](http://www4.ncsu.edu/~kksivara/publications/conic-ipm-decomposition.pdf)

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