MA796S / OR791K

SOLUTIONS TO HOMEWORK 4
PREPARED BY KARTIK

(1) Consider $F(x) = -\sum_{i=1}^{n} \log x_i$

We have $D_f = \{ x \in \mathbb{R}^n \mid x > 0 \}$

Consider any $x \in D_f$ and y satisfying

$$\|y-x\|_x < 1$$

We have $\|y-x\|_x^2 = \langle y-x, H(x)(y-x) \rangle$

$$= \langle y-x, X^{-2}(y-x) \rangle$$

where $X^{-2} = \begin{bmatrix} 1/x_1^2 & & 0 \\ & \ddots & \\ 0 & & 1/x_n^2 \end{bmatrix}$

$\therefore \|y-x\|_x^2 = \sum_{i=1}^{n} \left(\frac{y_i - x_i}{x_i} \right)^2 < 1$

$\therefore \quad -1 < \frac{y_i - x_i}{x_i} < 1$

$\therefore \quad 0 < y_i < 2x_i \qquad \therefore y \in D_f$

Now consider

$\|v\|_y^2 = \langle v, H(y) v \rangle = \sum_{j=1}^{n} \left(\frac{v_j}{y_j} \right)^2 = \sum_{j=1}^{n} \left(\frac{v_j}{x_j} \right)^2 \left(\frac{x_j}{y_j} \right)^2$

(2)

$$\therefore \|v\|_y^2 = \sum_{j=1}^{n} \left(\frac{v_j}{x_j}\right)^2 \left(\frac{x_j}{y_j}\right)^2$$

$$\leq \max_j \left(\frac{x_j}{y_j}\right)^2 \left(\sum_{j=1}^{n}\left(\frac{v_j}{x_j}\right)^2\right)$$

$$= \max_j \left(\frac{x_j}{y_j}\right)^2 \|v\|_x^2$$

$$\therefore \frac{\|v\|_y^2}{\|v\|_x^2} \leq \max_j \left(\frac{x_j}{y_j}\right)^2 \longrightarrow (1)$$

We have $\quad \dfrac{y_j}{x_j} \geq 1 - \left|\dfrac{y_j}{x_j} - 1\right|$

$$\geq 1 - \sum_{i=1}^{n}\left(\frac{y_j - x_j}{x_j}\right)^2$$

$$= 1 - \|y-x\|_x \quad \text{for all } j = 1,2,\dots,n$$

$$\therefore \frac{x_j}{y_j} \leq \frac{1}{1-\|y-x\|_x} \quad \text{for all } j=1,2,\dots,n \longrightarrow (2)$$

Using (1) and (2) we have

$$\frac{\|v\|_y^2}{\|v\|_x^2} \leq \frac{1}{(1-\|y-x\|_x)^2}$$

$$\text{i.e.} \quad \frac{\|v\|_y}{\|v\|_x} \leq \frac{1}{(1-\|y-x\|_x)}$$

SIMILARLY WE HAVE

$$\frac{\|v\|_y^2}{\|v\|_x^2} \geq \min_j \left(\frac{x_j}{y_j}\right)^2 = \frac{1}{\max_j \left(\frac{y_j}{x_j}\right)^2}$$

Now consider

$$\frac{y_j}{x_j} = \left|\frac{y_j}{x_j} + 1 - 1\right| \leq 1 + \left|\frac{y_j}{x_j} - 1\right|$$

$$\leq 1 + \|y-x\|_x$$

$$\therefore \frac{1}{\left(\frac{y_j}{x_j}\right)} \geq \frac{1}{1+\|y-x\|_x} \geq 1 - \|y-x\|_x$$

$\forall j = 1, 2, \ldots, n$

(since $\|y-x\|_x < 1$ and $\frac{1}{1+x} = 1 - x + x^2 + \cdots \geq (1-x)$ when $|x| < 1$)

$$\therefore \frac{1}{\max_j \left(\frac{y_j}{x_j}\right)^2} \geq \left(1 - \|y-x\|_x\right)^2$$

$$\therefore \frac{\|v\|_y^2}{\|v\|_x^2} \geq \left(1 - \|y-x\|_x\right)^2$$

$$\therefore \frac{\|v\|_y}{\|v\|_x} \geq \left(1 - \|y-x\|_x\right)$$

$$\therefore \quad 1-\|y-x\|_x \le \frac{\|a\|_y}{\|a\|_x} \le \frac{1}{(1-\|y-x\|_x)}$$

Do the SDP case as discussed in class notes.

(5)

(2) $$F(x) = -\log(x_1^2 - x_2^2 - \cdots - x_n^2)$$
$$= -\log(x^T J x)$$

where $$J = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{pmatrix}$$

$$g(x) = -\frac{1}{(x_1^2 - x_2^2 \cdots - x_n^2)} \begin{bmatrix} 2x_1 \\ -2x_2 \\ \vdots \\ -2x_n \end{bmatrix}$$

$$= \frac{-2}{x^T J x} \begin{bmatrix} x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix} = \frac{-2}{x^T J x} Jx$$

$$h(x) = -2 \left[\frac{(x^T J x) J - 2(J x x^T J)}{(x^T J x)^2} \right]$$ (using the quotient rule for derivatives)

$$= \frac{4}{(x^T J x)^2} J x x^T J - \frac{2}{x^T J x} J$$

2(b)

$$\theta(F) = \sup_{x \in D_f} \left(g(x)^T h(x)^{-1} g(x) \right)$$

We have

$$g(x)^T h(x)^{-1} g(x) = \left(\frac{-2}{x^T J x}\right)(Jx)^T \left(\frac{4}{(x^T J x)^2} J x x^T J - \frac{2}{x^T J x} J\right)^{-1} \left(\frac{-2}{x^T J x}\right)(Jx)$$

$$= (Jx)^T \left(\frac{2}{(x^T J x)} J x x^T J - J\right)^{-1} \left(\frac{2}{x^T J x}\right)(Jx)$$

$$= \frac{2}{(x^T J x)} x^T J^T \left(\frac{2}{(x^T J x)} J x x^T J - J\right)^{-1} (Jx)$$

$$= \frac{2}{(x^T J x)} x^T \left(\frac{2}{x^T J x} x x^T - J\right)^{-1} x$$

$$= \frac{2}{(x^T J x)} x^T \left[(-J) + \frac{2 J x x^T J}{x^T J x} \right] x$$

$\downarrow$

USING SHERMAN-MORRISON FORMULA $\quad (X + u v^T)^{-1} = \left(X^{-1} - \frac{X^{-1} u v^T X^{-1}}{1 + v^T X^{-1} u} \right)$

⑦

$$= \frac{2}{x^TJx}(-x^TJx) + 4\frac{x^TJx}{(x^TJx)^2}$$

$$= -2+4 = 2$$

$$\therefore \quad \theta(f) = \sup_{x \in D_f} 2 = 2$$

(3) SEE CLASS NOTES

(4) SEE CLASS NOTES

(5) WORK IT OUT YOURSELF.