MA/OR/ST 706: Nonlinear Programming
Class Project
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1 Introduction

Consider the portfolio optimization problem mentioned in Example 16.1 on pages 449-450 of Nocedal and Wright. We also considered this problem in class at the beginning of the course.

Here is a brief overview of the problem again: Suppose we have n possible investments with random returns r_i, $i = 1, \ldots, n$. Let $\mu_i = E[r_i]$ and $\sigma_i^2 = E[(r_i - \mu_i)^2]$ denote the expected value and the variance for the return on the ith investment. Let $\mu = [\mu_1, \ldots, \mu_n]$ be the vector of expected returns and Σ be the $n \times n$ symmetric covariance matrix with $\Sigma_{ii} = \sigma_i^2$ and $\Sigma_{ij} = E[(r_i - \mu_i)(r_j - \mu_j)]$ for $i \neq j$.

An investor constructs a portfolio by putting a fraction x_i of his/her available funds into the ith investment, for $i = 1, \ldots, n$. The objective of the investor is to construct the least risky portfolio, i.e., one that minimizes $x^T \Sigma x$ while guaranteeing a target return of R. The investor's program is the following quadratic program (QP)

$$\begin{array}{lrcl} \min & \frac{1}{2} x^T \Sigma x & & \\ \text{s.t.} & \mu^T x & \geq & R \\ & \sum_{i=1}^n x_i & = & 1 \\ & x_i & \geq & 0, \;\; i = 1, \ldots, n \end{array} \tag{1}$$

that is also known as Markowitz's mean-variance optimization (MVO) problem. The 2nd constraint $\sum_{i=1}^n x_i = 1$ indicates that all the available funds have been invested, while the 3rd set of constraints $x \geq 0$ ensures that no short selling is allowed.

We call a portfolio x *efficient* if it has the minimum variance (risk) among all the portfolios that have at least a certain expected return. The collection of all efficient portfolios forms the *efficient frontier*. Our objective in this project is to calculate this so-called efficient frontier which is plot of R versus the standard deviation.

This is done by solving the MVO problem (1) for ranges of R ranging from $R_{\min}$ and $R_{\max}$ and then plotting the expected return R versus the square root of the optimal objective value of (1). We will employ the active set algorithm (Algorithm 16.3) that is discussed in Section 16.5 of Chapter 16 of Nocedal and Wright to solve the (1) for varying values of R.

2 Project setup

Consider the quadratic program

$$
\begin{array}{lrcl}
\min & 0.02778x_s^2 + 0.000774x_S x_B + 0.00042x_S x_M & & \\
 & +0.01112x_B^2 - 0.00040x_B x_M + 0.00115x_M^2 & & \\
\text{s.t.} & 0.1073x_S + 0.0737x_B + 0.0627x_M & \geq & R \\
 & x_S + x_B + x_M & = & 1 \\
 & x_S & \geq & 0 \\
 & x_B & \geq & 0 \\
 & x_M & \geq & 0
\end{array} \qquad (2)
$$

where x_S, x_B, and x_M denote the proportion of the capital invested in stocks, bonds, and money market savings accounts, respectively. The QP (2) is derived from the data in Table 8.1 in Section 8.1.1 (pages 140-143) of Cornuejols and Tütüncü [1], and is problem (8.6) in this book. You can download Section 8.1 of Cornuejols and Tütüncü from the course webpage. Please read the discussion in section 8.1.1 to see how the QP is set up from the data in Table 8.1.

We will solve (2) for values of R ranging between $R_{\min} = 6.5\%$ to $R_{\max} = 10.5\%$ in increments of 0.5% using the active set algorithm (Algorithm 16.3) on page 472 of Nocedal and Wright. The optimal objective value of the QP for a given R gives the variance of the portfolio. Taking the positive square root of this value will give the standard deviation (risk) of the portfolio. Please read Section 16.5 in Nocedal and Wright carefully before writing your computer program in MATLAB. To obtain the efficient frontier, you will plot the values of R versus the corresponding standard deviations. Your efficient frontier for this problem should resemble the plot in the left hand side of Figure 8.1 in Cornuejols and Tütüncü.

An important feature of the active set algorithm is that it is conducive to warm-starts, and we can exploit this feature while solving the family of quadratic programs for various values of R. For instance, we can solve the QP (2) with $R = 10.5\%$ from scratch to obtain the solution x^1 (say). When we solve QP again for $R = 10\%$, we can use the earlier optimal solution x^1 as a starting point. Why?

3 Details of the project

1. You will work in a group consisting of 1-2 students. The entire project is due in class on Thursday, the 25th of April.

2. Read Sections 16.1-16.2, 16.4-16.5 in Nocedal and Wright carefully before beginning the project. Other references for the project include the material in section 8.1 of Cornuejols and Tütüncü [1] that can be downloaded from the course webpage.

3. Each group will submit a 4-5 page report that describes the portfolio optimization problem, the details of the active set algorithm in obtaining the efficient frontier, and your computational experiences with the algorithm on the quadratic programs discussed in part (6). mentioned in section 2 and exercises 8.3 and 8.4 on page 144 of Cornuejols and Tutuncu.

4. You will write a computer program that implements Algorithm 16.3 (active set algorithm) in Nocedal and Wright in MATLAB and returns an efficient frontier for the portfolio optimization problem. The active set algorithm will take n (the number of investments), Σ (the $n \times n$ covariance matrix), μ (the vector of expected returns), a starting solution x as input, and an optimality tolerance $\epsilon = 10^{-6}$. The program will return x (the vector of investments), the optimal objective value, and the number of iterations needed by the algorithm. In each iteration of the algorithm, we have to solve an equality constrained quadratic subproblem (16.39) in order to compute the search direction p_k. You will use the *Schur complement method* in Section 16.2 of Nocedal and Wright to solve the KKT system (see Section 16.1) arising from the equality constrained quadratic program.

5. Test your algorithm in part (4) on the quadratic program discussed in section 2. You will also run your algorithm on Exercises 8.3 and 8.4 on page 144 of Cornuejols and Tütüncü. For Exercise 8.4, you will follow the procedure in section 8.1.1 (pages 140-143) to construct the expected return vector μ, and the covariance matrix Σ. You should include your solutions and the efficient frontier for these examples along with a brief summary of your findings in the report.

6. You should use MATLAB's quadratic programming solver *quadprog* to verify your efficient frontiers for the three quadratic programs that you considered in part (5). Type *help quadprog* at the MATLAB prompt for details on how to use *quadprog*.

References

[1] G. Cornuejols and R. Tütüncü, *Optimization Methods in Finance*, Cambridge University Press, 2007.