MA/OR/ST 706: Nonlinear Programming
Homework 5
Instructor: *Dr. Kartik Sivaramakrishnan*

INSTRUCTIONS

Due in class on Tuesday April 8, 2008. Please read your class notes and Chapter 12 in Nocedal and Wright before beginning the assignment. No late homeworks will be accepted without prior instructor approval.

1. Solve the problem

$$\begin{array}{rrcl} \min & \displaystyle\sum_{i=1}^n x_i^2 & & \\ \text{s.t.} & \displaystyle\sum_{i=1}^n x_i & = & 1 \\ & x & \geq & 0 \\ & x & \leq & u \end{array} \tag{1}$$

   where $x, u \in I\!R^n$. **Hint:** Consider the dual problem.

2. Consider the portfolio optimization

$$\begin{array}{rrcl} \min & x^T V x & & \\ \text{s.t.} & \mu^T x & \geq & R, \\ & \displaystyle\sum_{i=1}^n x_i & = & 1, \\ & x_i & \geq & 0, \ \ i = 1, \ldots, n \end{array} \tag{2}$$

   where V is a $n \times n$ symmetric positive definite matrix, $\mu \in I\!R^n$, and R is a positive scalar. The optimization variable is $x \in I\!R^n$. We considered this problem at the beginning of the course and also in the class project. Write down the dual problem.

3. Consider the following quadratic program

$$\begin{array}{lrcl} \displaystyle\min_{v,\gamma,s,z} & \displaystyle\sum_{i=1}^k s_i + \sum_{j=1}^m z_j + \delta ||v||^2 & & \\ \text{s.t.} & a_i^T v - \gamma + s_i & \geq & 1, \ \ i = 1, \ldots, k \\ & b_j^T v - \gamma - z_j & \leq & -1, \ \ j = 1, \ldots, m \\ & s & \geq & 0 \\ & z & \geq & 0 \end{array} \tag{3}$$

where the optimization variables are $v \in I\!R^n$, $\gamma \in I\!R$, $s \in I\!R^k$, and $z \in I\!R^m$. The problem rises in the classification (support vector machines) example that we considered in the beginning of the course. Write down the dual problem.

4. A model represents the output variable, $y \in I\!R$, as a linear function

$$y = \sum_{i=1}^{n} x_i u_i,$$

of input variables $u_1, \ldots, u_n$. The quantities $x_1, \ldots, x_n$ are unknown model coefficients. We have m observations of input and output variables: (u^j, y^j), $j = 1, \ldots, m$. One way to determine the values of the coefficients $x_1, \ldots, x_n$ is to minimize the maximum absolute error:

$$f(x) = \max_{j=1,\ldots,m} |y^j - \sum_{i=1}^{n} x_i u_i^j|.$$

This problem can be written as the following constrained optimization problem

$$\begin{array}{lrcl} \min & & v & \\ \text{s.t.} & y^j - \sum_{i=1}^{n} x_i u_i^j & \geq & -v, \;\; j = 1, \ldots, m \\ & y^j - \sum_{i=1}^{n} x_i u_i^j & \leq & v, \;\; j = 1, \ldots, m. \end{array} \tag{4}$$

Formulate and analyze the dual problem to (4).

5. Consider the maxcut problem that we discussed in class. This problem can be formulated as the following quadratic program with integer constraints

$$\begin{array}{lrcl} \min & x^T Q x & & \\ \text{s.t.} & x_i^2 & = & 1, \;\; i = 1, \ldots, n \end{array} \tag{5}$$

where $x \in I\!R^n$ and Q is a symmetric $n \times n$ matrix (not necessarily positive semidefinite). We showed in Homework 4 that the Lagrangian dual problem is given by the following semidefinite program (SDP)

$$\begin{array}{lrcl} \max & e^T y & & \\ \text{s.t.} & Q - \text{Diag}(y) & \succeq & 0 \end{array} \tag{6}$$

where $y \in I\!R^n$ and $\text{Diag}(y)$ is the diagonal matrix of size n with the components of y along the diagonal. The optimal objective value of the SDP provides a lower bound on the optimal objective value of the maxcut problem. In this exercise, we derive another SDP that gives a lower bound on the optimal objective value of the maxcut problem, and also explore the connection between the two SDPs.

(a) Show that the maxcut problem can also be written as

$$\begin{array}{lrcl} \min & Q \bullet X & & \\ \text{s.t.} & X_{ii} & = & 1, \;\; i = 1, \ldots, n \\ & X & \succeq & 0, \\ & \operatorname{rank}(X) & = & 1 \end{array} \tag{7}$$

where X is a symmetric matrix of size n, $Q \bullet X = \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij} X_{ij}$ is the dot product of two symmetric matrices, $X \succeq 0$ indicates that the matrix X is positive semidefinite, and $\operatorname{rank}(X)$ is the rank of the symmetric positive semidefinite matrix X (number of positive eigenvalues).

Hint: Show that any x that is feasible in (5) gives a matrix $X = xx^T$ that is feasible in (7). Moreover, show that any X that is feasible in (7) has the form $X = xx^T$ where x is feasible in (5).

(b) Show that dropping the restriction $\operatorname{rank}(X) = 1$ gives the following SDP relaxation

$$\begin{array}{lrcl} \min & Q \bullet X & & \\ \text{s.t.} & X_{ii} & = & 1, \;\; i = 1, \ldots, n \\ & X & \succeq & 0 \end{array} \tag{8}$$

of the maxcut problem. This SDP can be solved efficiently. Explain why the optimal objective value of (8) provides a lower bound on the optimal objective value of (5). What can you say if an optimal solution X^* to (8) is rank one?

(c) We have two SDPs: (6) and (8) that provide a lower bound on the optimal objective value of the maxcut problem. Relate the two SDPs by duality. In particular, show that the Lagrangian dual of (8) is the other SDP (6). What can you say about the optimal objective values of these two SDPs? **Hint:** Show that the SDP (8) has a Slater point, i.e., there is a feasible $\bar{X}$ in (8) such that $\bar{X} \succ 0$ (positive definite).