

Nonlinear Programming Qualifying Exam

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INSTRUCTIONS

1. Please write your name and student number clearly on the front page of the exam.
2. This has to be your own work. Cheating on the exam is not tolerated, and will fetch you a zero for the test.
3. TIME LIMIT: 3 hours
4. There are 5 pages and 4 questions on the exam. Each question appears on a different page. Read each question carefully.
5. The exam is worth 110 points. The distribution of points is clearly indicated on the exam.
6. Solve each problem in sufficient detail in the space provided. Please use both sides of each page as needed.
7. Write clearly, including all the steps to the final solution. If I can't read it, you won't get credit.
8. Sources: Nocedal & Wright (2nd edition), class notes and homeworks, and solutions to homeworks and midterm exam.
9. You may use an electronic calculator on the exam.

1. [25 points] Consider a symmetric $n \times n$ matrix Q . Consider the problem

$$\begin{aligned} \min \quad & x^T Q x \\ \text{s.t.} \quad & x^T x = 1 \end{aligned}$$

and let $e_1 \in \mathbb{R}^n$ and $\lambda_1 \in \mathbb{R}$ be an optimal solution and the optimal objective value to the problem, respectively. Moreover, for $k = 2, \dots, n$, consider the problem

$$\begin{aligned} \min \quad & x^T Q x \\ \text{s.t.} \quad & x^T x = 1 \\ & e_i^T x = 0, \quad i = 1, \dots, k-1 \end{aligned}$$

and let $e_k \in \mathbb{R}^n$ and $\lambda_k \in \mathbb{R}$ be an optimal solution and the optimal objective value to the problem, respectively.

- (a) [5 points] Show that $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$.
- (b) [5 points] Show that the vectors e_1, \dots, e_n are linearly independent.
- (c) [15 points] Using the first order optimality conditions, show that $\lambda_1, \dots, \lambda_n$ are the eigenvalues of Q , while e_1, \dots, e_n are the corresponding eigenvectors.

2. [25 points] Consider the problem

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & Ax \leq b \\ & Ex = e. \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $E \in \mathbb{R}^{p \times n}$, $b \in \mathbb{R}^m$, $e \in \mathbb{R}^p$, and $f(x)$ is a twice continuously differentiable function. Suppose that \bar{x} is a feasible solution such that $A_1 \bar{x} = b_1$ and $A_2 \bar{x} < b_2$ where $A^T = [A_1^T \ A_2^T]$ and $b^T = [b_1^T \ b_2^T]$. Suppose that $M^T = [A_1^T \ E^T]$ has full column rank, consider the matrix

$$P = I - M^T(MM^T)^{-1}M.$$

Furthermore, suppose that $P\nabla f(\bar{x}) = 0$, and let $\bar{w} = -(MM^T)^{-1}M\nabla f(\bar{x})$ and $\bar{w}^T = [\bar{u}^T \ \bar{v}^T]$.

(a) [5 points] If $\bar{u} \geq 0$, show that \bar{x} is a KKT point of (1).

(b) [20 points] If $\bar{u} \not\geq 0$, let u_j be a negative component of \bar{u} , and let $\hat{M}^T = [\hat{A}_1^T \ E^T]$, where \hat{A}_1 is obtained from A_1 by deleting the row of A_1 corresponding to u_j . Moreover, let $\hat{P} = I - \hat{M}^T(\hat{M}\hat{M}^T)^{-1}\hat{M}$, and let $\hat{d} = -\hat{P}\nabla f(\bar{x})$. Show that \hat{d} is an improving direction, i.e., $\bar{x} + \lambda\hat{d}$ is feasible in (1) and $f(\bar{x} + \lambda\hat{d}) < f(\bar{x})$ for $\lambda > 0$ and sufficiently small.

Hint: Show that $\hat{d} \neq 0$.

3. [25 points] Consider the following convex quadratic program

$$\begin{aligned} \min \quad & \frac{1}{2}x^T Gx + c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{2}$$

where $c, x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and G is a symmetric positive semidefinite matrix of size n .

- (a) [5 points] Write the KKT conditions for (2).
- (b) [10 points] We will replace (2) with the following sequence of barrier subproblems

$$\begin{aligned} \min \quad & \frac{1}{2}x^T Gx + c^T x - \mu \sum_{i=1}^m \log x_i \\ \text{s.t.} \quad & Ax = b \end{aligned} \tag{3}$$

where $\mu > 0$ is an appropriate barrier parameter. What are the KKT conditions for (3)? Use the KKT conditions to derive an analogue of the generic primal-dual step (16.58 - see page 481 of Nocedal & Wright) for this problem.

- (c) [10 points] Suggest an efficient way to solve the linear system in part (b).
Hint: See equations (16.61) and (16.62) on page 482 of Nocedal & Wright.

4. [35 points] Consider the *binary linear program*

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x_i \in \{0, 1\}, \quad i = 1, \dots, n \end{aligned} \tag{4}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The binary LP is in general a very difficult problem to solve. One can instead solve the LP relaxation

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & 0 \leq x_i \leq 1, \quad i = 1, \dots, n \end{aligned} \tag{5}$$

which is far easier to solve and gives a lower bound on the optimal objective value of (4). In this problem, we will derive another lower bound for the binary LP using Lagrangian relaxation.

(a) [25 points] The boolean LP can be reformulated as

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x_i^2 = x_i, \quad i = 1, \dots, n \end{aligned} \tag{6}$$

which has quadratic equality constraints. Show that the Lagrangian dual to (6) can be written as

$$\max_{u \geq 0} -b^T u + \sum_{i=1}^n \max_{w_i} \theta_i(u, w_i) \tag{7}$$

where $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$ are the Lagrangian multipliers for the 1st and the 2nd set of constraints in (6) and

$$\begin{aligned} \theta_i(u, w_i) &= -\frac{((c + A^T u)_i - w_i)^2}{4w_i} && \text{if } w_i > 0, \\ &= 0 && \text{if } w_i = 0 \text{ and } (c + A^T u)_i = 0, \\ &= -\infty && \text{otherwise} \end{aligned}$$

where $(c + A^T u)_i$ is the i th component of the vector $(c + A^T u)$.

(b) [10 points] Show that the lower bound obtained by solving the Lagrangian relaxation (7) is the same as that obtained by solving the LP relaxation (5).

Hint: Show that $\max_{w_i} \theta_i(u, w_i) = \min\{0, (c + A^T u)_i\}$ and consider the dual of the LP relaxation (5).