

MA/OR/ST 706: Nonlinear Programming
Final Exam Review
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INSTRUCTIONS

The questions on the final will be similar to these review problems. Work out each problem in sufficient detail. If you have any questions, please send me an email. I will post the solutions to selected problems on the course webpage next Friday. All problems are from the 2nd edition of Nocedal and Wright unless otherwise specified.

1. Using the KKT conditions, show that the optimal solution

$$p_k^s = \arg \min_{p \in \mathbb{R}^n} f_k + g_k^T p \text{ s.t. } \|p\| \leq \Delta_k,$$

where $f_k \in \mathbb{R}$, $g_k \in \mathbb{R}^n$, and Δ_k is a positive scalar, is given by

$$p_k^s = -\frac{\Delta_k}{\|g_k\|} g_k.$$

We used this result in the calculation of the Cauchy point in Section 4.1 of Nocedal-Wright (see pages 71-72).

2. Consider the following problem

$$\begin{aligned} \min_x \quad & \sum_{j=1}^n \frac{c_j}{x_j} \\ \text{s.t.} \quad & \sum_{j=1}^n a_j x_j = b \\ & x_j \geq 0, \quad j = 1, \dots, n \end{aligned}$$

where a_j , b , and c_j are positive constants. Write down the KKT conditions for this problem and solve for the point $x^* \in \mathbb{R}^n$ that satisfies these conditions.

3. Consider the problem

$$\begin{aligned} \min \quad & \sum_{j=1}^n x_j \\ \text{s.t.} \quad & \prod_{j=1}^n x_j = 1, \\ & x_j \geq 0, \quad j = 1, \dots, n. \end{aligned}$$

- (a) What are the KKT conditions for the problem?
- (b) Using the KKT conditions, find an optimal solution to the problem.
- (c) Use the optimal solution and the corresponding optimal objective value to show the following inequality: If $x_1, \dots, x_n \geq 0$, then

$$\frac{1}{n} \sum_{j=1}^n x_j \geq \left(\prod_{j=1}^n x_j \right)^{\frac{1}{n}}.$$

4. Consider the problem

$$\begin{aligned} \max \quad & x_1^2 + 4x_1x_2 + x_2^2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 = 1. \end{aligned}$$

(Note that the problem is a maximization problem).

- (a) Using the KKT conditions, find an optimal solution to the problem.
 - (b) Test for the second order optimality conditions.
 - (c) Does the problem have a unique optimal solution? Why or why not?
5. Consider the problem

$$\begin{aligned} \min \quad & e^{-2x} \\ \text{s.t.} \quad & x \geq 0 \end{aligned}$$

in primal form.

- (a) Solve the primal problem.
 - (b) What is the Lagrangian dual problem? Solve the dual problem.
6. Consider the convex optimization problem

$$\min_x \frac{1}{2} \|x - z\|^2 + f(x)$$

where

$$f(x) = \max_{1 \leq i \leq m} (a_i^T x + b_i).$$

The vectors $z \in \mathbb{R}^n$, $a_i \in \mathbb{R}^n$, and the scalars b_i are fixed.

- (a) Show that the above problem is equivalent to the following quadratic programming problem by introducing a new variable $v \in \mathbb{R}$:

$$\begin{aligned} \min_{x,v} \quad & \frac{1}{2} \|x - z\|^2 + v \\ \text{s.t.} \quad & v \geq a_i^T x + b_i, \quad i = 1, \dots, m. \end{aligned}$$

- (b) Formulate the dual problem.

7. Consider the linear programming problem

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array}$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and the optimization variable $x \in \mathbb{R}^n$. Formulate the Lagrangian dual problem.

8. Consider the projection problem

$$\begin{array}{ll} \min_x & \frac{1}{2} \|x - z\|^2 \\ \text{s.t.} & Ax = 0 \end{array}$$

where $z \in \mathbb{R}^n$ is fixed and $A \in \mathbb{R}^{m \times n}$. Write down the Lagrangian dual problem.

9. Problem 16.25, Page 495.

10. Problem 17.9, Page 528.