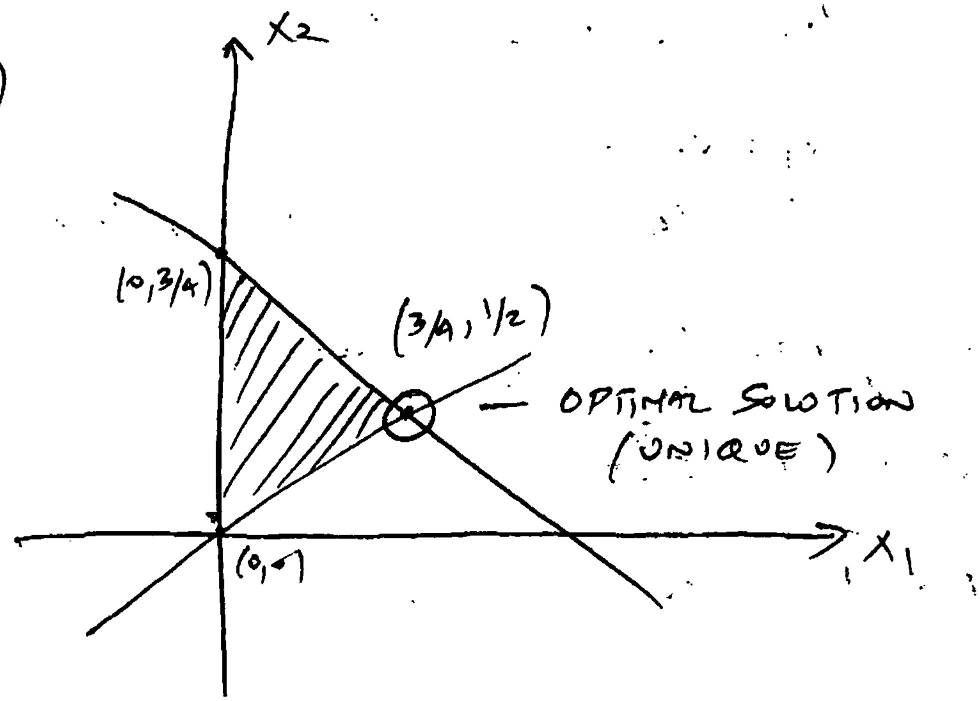
MA 505-001 LINEAR PROGRAMMING
SOLUTIONS TO HOMEWORK 4
(PREPARED BY KARTIK)

(1) DO PROBLEM (1) YOURSELF. CHECK YOUR ANSWERS WITH KARTIK'S CODE POSTED ON WEBPAGE

(2) Consider

$$z = \text{Max } x_1$$
$$\text{s.t. } x_1 + 3x_2 \le 9/4$$
$$2x_1 - 3x_2 \le 0$$
$$x \in X = \left\{ (x_1, x_2) \;\middle|\; \begin{array}{l} 0 \le x_1 \le 1 \\ 0 \le x_2 \le 1 \end{array} \right\}$$

(a)



You can solve the problem using Kartik's revised simplex code from the course webpage. We can take the origin as the starting point.

(2)

Note that the origin is a DEGENERATE extreme point. The revised simplex method takes 2 simplex iterations to find the optimal solution $x^* = (\frac{3}{4}, 1/2)$.

(The first iteration is a degenerate PIVOT, i.e. we continue to stay at the origin $(0,0)$)

(b) Rewrite the problem as

$$\text{Max} \quad x_1$$

$$\text{s.t.} \quad x_1 + 3x_2 + x_5 = \frac{9}{4}$$

$$2x_1 - 3x_2 + x_6 = 0$$

$$x_1 + x_3 = 1$$

$$x_2 + x_4 = 1 \qquad x_1, \ldots, x_6 \geq 0$$

This problem has the structure

$$\text{Max} \quad c^T x + d^T \omega$$

$$\text{s.t} \quad Ax + D\omega = b$$

$$B_1 x = d_1$$

$$x \geq 0, \quad \omega \geq 0$$

where

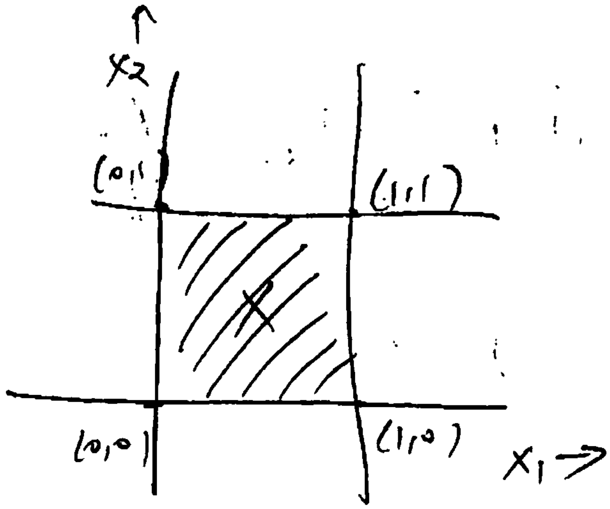
$A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 2 & -3 & 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 9/4 \\ 0 \end{bmatrix}$

$d_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad w = \begin{bmatrix} x_5 \\ x_6 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and $d = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Let $X = \left\{ \cancel{x} = (x_1, x_2, x_3, x_4) \;\middle|\; \begin{array}{l} x_1 + x_3 = 1 \\ x_2 + x_4 = 1 \\ x_1, x_2, x_3, x_4 \geq 0 \end{array} \right\}$



The extreme points of

$$X = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Note that X has a very special structure, we can write down its extreme points by inspection.

Since X has such an easy characterization of extreme points, we treat its constraints as the constraints in the subproblem.

We solve this problem using the code from Problem 1 part b) that I have posted on the course webpage (PLEASE RUN THIS CODE FOR SOLUTION DETAILS)

The master problem is

$$\text{Max} \sum_{j \in J_1} (c^T x^j)\lambda^j + d^T w$$

$$\text{s.t} \sum_{j \in J_1} \begin{bmatrix} Ax^j \\ 1 \end{bmatrix} \lambda^j + \begin{bmatrix} D \\ 0 \end{bmatrix} w = \begin{bmatrix} b \\ 1 \end{bmatrix}$$

$$\lambda^j \geq 0 \quad j \in J_1$$
$$w \geq 0$$

The subproblem is

$$\text{Max} \quad (c - A^T y)^T x$$

$$\text{s.t} \quad B_1 x = d_1$$
$$x \geq 0$$

The optimal solution is

$$x_B = \begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix} \qquad A_B = \begin{bmatrix} 0 & 4 & 2 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$c_B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(5)

The reduced costs at optimality

are $s_N = \begin{bmatrix} 0 \\ -1/3 \\ -1/3 \end{bmatrix}$ indicating that there are MULTIPLE optimal solutions for the master problem

We needed 3 iterations with my decomposition code to solve the problem.

c) The original problem has a unique solution $x^* = \left(\frac{3}{4}, 1/2\right)$

The reformulated master problem has MULTIPLE optimal solutions

Thus, the reformulation ALTERS the combinatorial structure of the original problem.

(3) Consider the LP

$$\text{Max } c^T x + d^T y$$
$$\text{s.t } \quad Ax + Dy \le b$$
$$Fx \le f$$
$$x, y \ge 0$$

We can treat this problem in 2 ways

(a) Coupling / Complicating constraint $Ax + Dy \le b$

In this case, we use the Dantzig-Wolfe decomposition scheme and our subproblem has the form

$$\text{Max } (c - A^T w)^T x$$
$$\text{s.t } \quad Fx \le f$$
$$x \ge 0$$

where w is the vector of dual variables for $Ax + Dy \le b$ (SEE CLASS NOTES)

Since we have access to a fast subroutine to

Solve $\quad \text{Max } h^T x$
$\quad \text{s.t } \quad Fx \le f$

and the subproblem is in the form, we will use the Dantzig-Wolfe (DW) decomposition scheme.

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(b) We can also treat x as a vector of complicating / COUPLING variables in the primal problem.

In this case, we apply Benders' decomposition to solve the LP, or the Dantzig-Wolfe (D-W) Scheme to solve the dual problem.

The dual LP is

$$\begin{aligned} \text{MIN} \quad & b^T w + f^T z \\ \text{s.t} \quad & A^T w + F^T z \geq c \\ & D^T w \geq d \\ & w, z \geq 0 \end{aligned}$$

where w is the vector of dual variables corresponding to $Ax + Dy \leq b$ etc.

Note that this LP is also in the block-angular form where $A^T w + F^T z \geq c$ are the COUPLING constraints

If we apply the D-W Scheme to this problem our subproblems will be

$$\begin{aligned} \text{MIN} \quad & (b - Ax)^T w \\ \text{s.t} \quad & D^T w \geq d \\ & w \geq 0 \end{aligned}$$

hence, we can apply Benders' decomposition on the original LP

Since we have access to a fast subroutine to solve

$$\begin{aligned} \text{Max} \quad & d^T y \\ \text{s.t} \quad & Dy \leq h \\ & y \geq 0 \end{aligned}$$

we can solve the subproblem efficiently too.

(NOTE that the subproblem is the DUAL to the fast subroutine problem)

4. Consider

$$Z = \text{Max} \quad 9x_1 + 5x_2$$

$$\begin{aligned} \text{s.t.} \quad 4x_1 + 9x_2 &\le 35 \\ x_1 &\le 6 \\ -(x_1 - 3x_2) &\le -1 \\ 3x_1 + 2x_2 &\le 19 \\ x_1 \geq 0, \quad x_2 &\geq 0 \\ x_1 \text{ integer}, \quad x_2 &\geq 0 \end{aligned} \quad \rightarrow \quad 1$$

Since the origin is not feasible in the LP relaxation of 1 we will use the phase 1 procedure to generate an initial feasible point

The auxiliary problem is

$$\begin{aligned} \text{MAX} \quad & -x_0 \\ \text{s.t.} \quad & 4x_1 + 9x_2 + x_3 - x_0 = 35 \\ & x_1 + x_4 - x_0 = 6 \\ & -x_1 + 3x_2 + x_5 - x_0 = -1 \\ & 3x_1 + 2x_2 + x_6 - x_0 = 19 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \\ & x_0 \geq 0 \end{aligned} \quad \rightarrow \quad 2$$

An initial feasible pt for 2 is

$x_1 = 0$, $x_2 = 0$, $x_0 = 1$, $x_3 = (35 + x_0) = 36$, $x_4 = (6 + x_0) = 7$, $x_5 = 0$, $x_6 = (19 + x_0) = 20$

In 2 simplex iterations with Karthik's revised simplex we get the following phase 1 optimal solution

$$x = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 5 \\ 0 \\ 16 \\ 0 \end{pmatrix}$$

This gives the feasible solution

$$x = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 5 \\ 0 \\ 16 \end{pmatrix}$$ for the original LP (1)

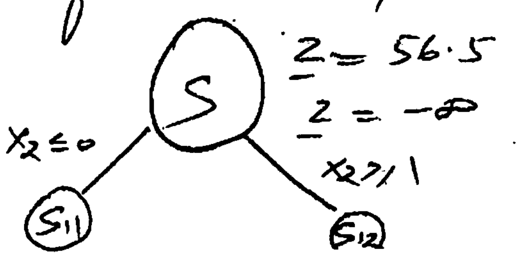
In 3 revised simplex iterations with Karthik's revised simplex code we get the optimal solution to (1) which is

$$x^* = \begin{pmatrix} 6 \\ 1/2 \\ 13/2 \\ 0 \\ 7/2 \\ 0 \end{pmatrix}$$

This gives the initial upper bound $\bar{z} = 56.5$

The current lower bound $\underline{z} = -\infty$

We will branch on x_2 (which is the most fractional) original variable



We solve problem S[redacted] using the dual simplex method. The new constraint is $x_2 + x_7 = 0$

Feed the data

$$A = \begin{bmatrix} 4 & 9 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 35 \\ 6 \\ -1 \\ 19 \\ 0 \end{bmatrix} \quad c = \begin{bmatrix} 9 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and $$x_0 = \begin{bmatrix} 6 \\ 1/2 \\ 13/2 \\ 0 \\ 7/2 \\ 0 \\ -1/2 \end{bmatrix}$$

We need two dual simplex iterations and the optimal solution is

$$x^* = \begin{bmatrix} 6 \\ 0 \\ 11 \\ 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}$$ for an optimal objective value of 54

$\therefore \quad \bar{z} = 54 \quad$ and $\quad \underline{z} = 54$

The subproblem S[redacted] is fathomed (pruned) by optimality

Now consider the subproblem S_2 :
The new constraint is

Feed the data $x_2 - x_7 = 1$

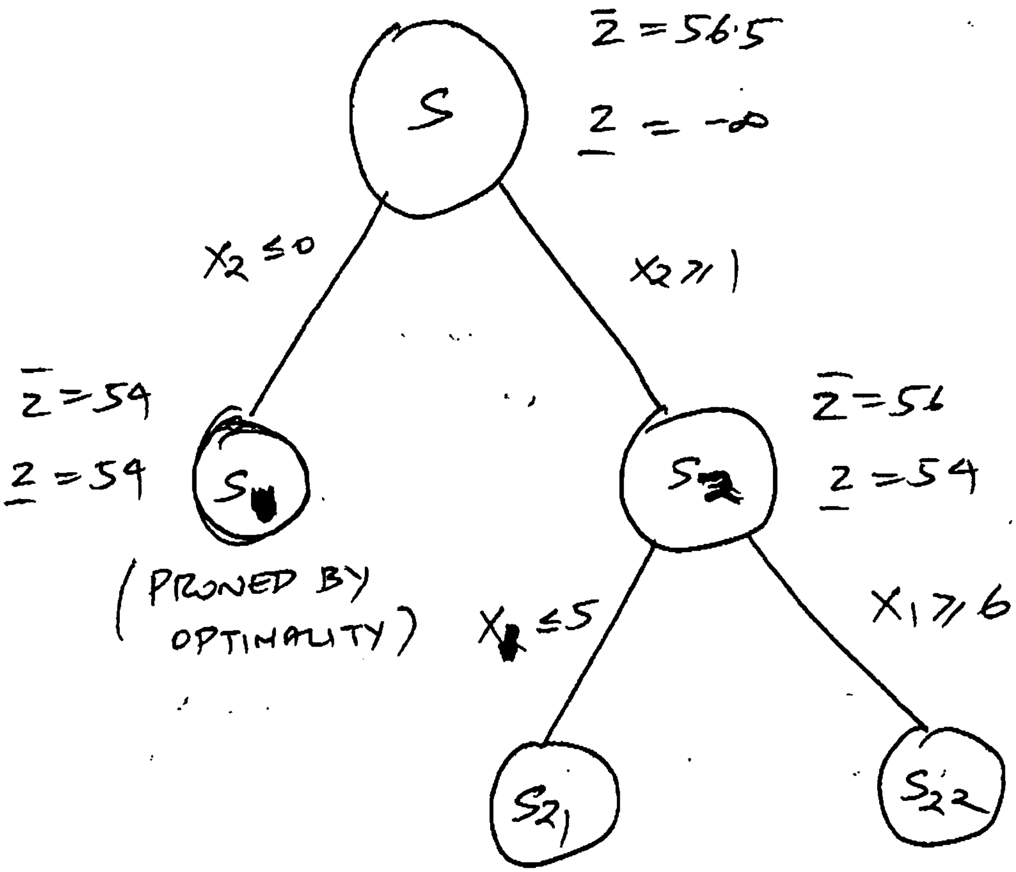
$$A = \begin{bmatrix} 9 & 9 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 35 \\ 6 \\ -1 \\ 19 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 9 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and $$x^0 = \begin{bmatrix} 6 \\ 1/2 \\ 13/2 \\ 0 \\ 7/2 \\ 0 \\ -1/2 \end{bmatrix}$$

We need two dual simplex iterations and we get the optimal solution

$$x^* = \begin{bmatrix} 17/3 \\ 1 \\ 10/3 \\ 1/3 \\ 5/3 \\ 0 \\ 0 \end{bmatrix}$$ for a objective value of $\bar{z} = 56$

We branch on x_1
creating two subproblems
S_{21} and S_{22}
$(x_1 \le 5)$ $(x_1 \ge 6)$



Consider S_{21} first. The new constraint is
$x_1 + x_8 = 5$

Feed the data

$$A = \begin{bmatrix} 4 & 9 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 35 \\ 6 \\ -1 \\ 19 \\ 1 \\ 5 \end{bmatrix}$$

$$c = \begin{bmatrix} 9 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and $x_0 = \begin{bmatrix} 17/3 \\ 1 \\ 10/3 \\ 1/3 \\ 5/3 \\ 0 \\ -2/3 \end{bmatrix}$ to Kartik's dual simplex code

We get the optimal solution after 4 dual simplex iterations

that is

$$x^* = \begin{bmatrix} 5 \\ 4/3 \\ 3 \\ 1 \\ 0 \\ 4/3 \\ 1/3 \\ 0 \end{bmatrix}$$ for an objective value of

$$\bar{z} = 51.67$$

Since $\bar{z} = 51.67 < \underline{z} = 54$, this node is pruned by bounds.

Consider the problem S_{22}. The new constraint is $x_1 - x_8 = 6$

Feed the data

$$A = \begin{bmatrix} 4 & 9 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 35 \\ 6 \\ -1 \\ 19 \\ 1 \\ 6 \end{bmatrix}, \quad c = \begin{bmatrix} 9 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and $$x_0 = \begin{bmatrix} 17/3 \\ 1 \\ 10/3 \\ 1/3 \\ 5/3 \\ 0 \\ 0 \\ -1/3 \end{bmatrix}$$

This problem is infeasible and is pruned by infeasibility.

This gives the optimal solution

$x^* = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$ for an objective value of $z^* = 54$

3 Consider the problem

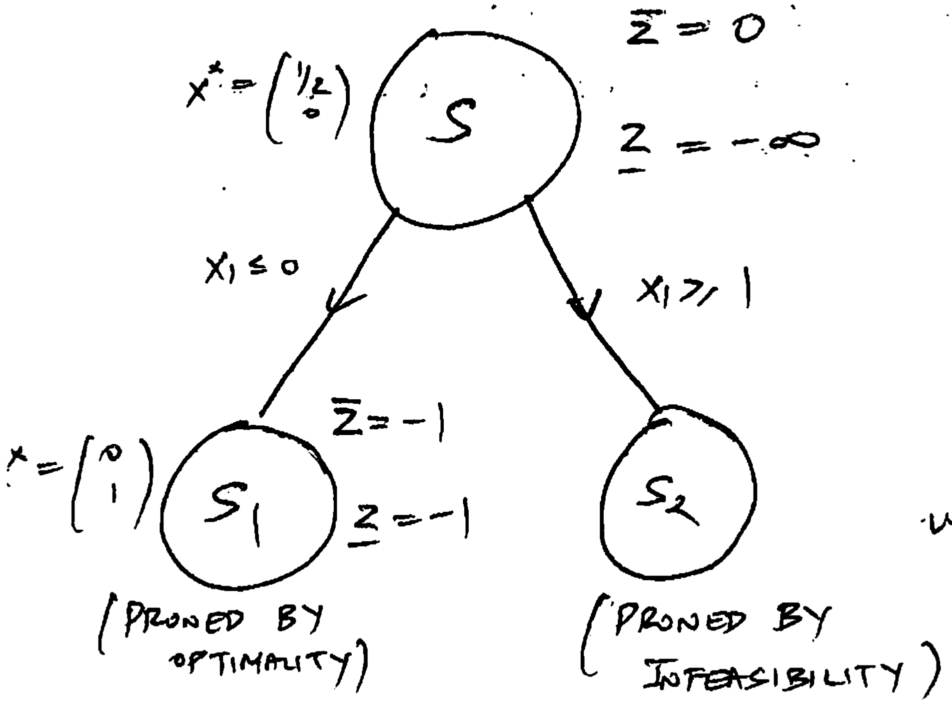
$$\begin{aligned} \text{Max} \quad & -x_{n+1} \\ \text{s.t.} \quad & 2x_1 + 2x_2 + \cdots + 2x_n + x_{n+1} = n \\ & x_i \in \{0,1\}, \quad i = 1, 2, \ldots, n+1 \end{aligned}$$

Assume $n = 1$

Our problem is

$$\begin{aligned} \text{Max} \quad & -x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 = 1 \\ & x_1 \in \{0,1\} \\ & x_2 \in \{0,1\} \end{aligned}$$

Our B & B tree is



We need 3 B & B nodes to find and verify that (0,1) is the OPTIMAL SOLUTION.

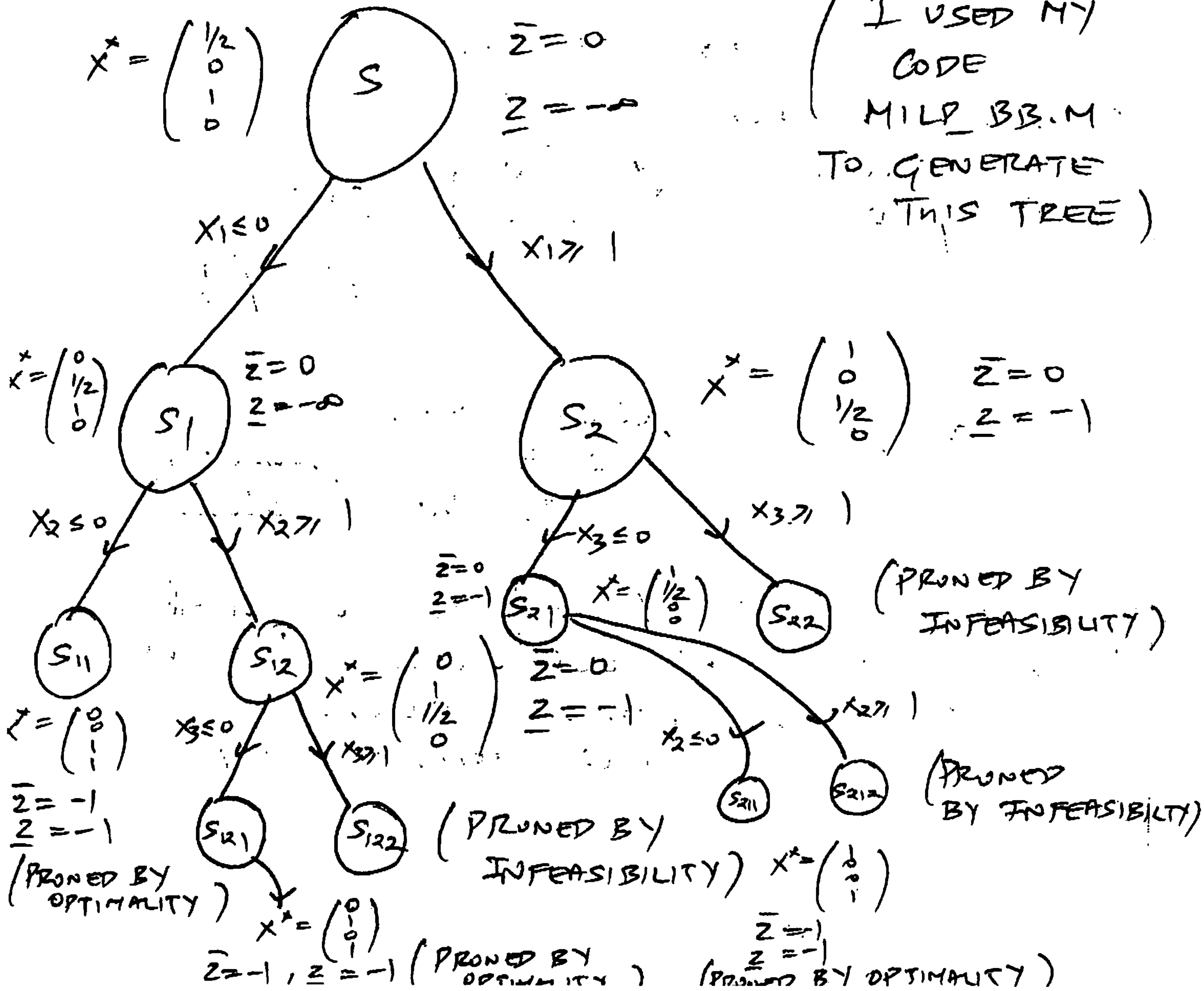
Consider $n=3$.

Our problem is

$$\begin{aligned} &\text{Max} \quad -x_4 \\ &\text{s.t.} \quad 2x_1 + 2x_2 + 2x_3 + x_4 = 3 \\ &\qquad x_i \in \{0,1\}, \quad i = 1,2,3,4 \end{aligned}$$

The B&B tree is

(I USED MY CODE MILP_BB.M TO GENERATE THIS TREE)



It is clear that the size of the B&B grows exponentially with n (LIKE $2^n + n$)

The problem is that we are unable to prune any of the nodes using BOUNDS (they are pruned only by optimality or by infeasibility).

Along the way, we generate 3 optimal solutions

$$x^* = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad x^* = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } x^* = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

respectively

(These are due to the symmetry in the problem)

So it is easy to find these optimal solutions but the certificate that we did find an optimal solution takes FOREVER :-(