Major/IE 505 Linear Programming

Solutions to Homework 2
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(1) \( z = \text{Max} \quad 7x_1 + 6x_2 + 5x_3 - 2x_4 + 3x_5 \)

\[ \begin{align*}
5: & \quad x_1 + 3x_2 + 5x_3 - 2x_4 + 2x_5 \leq 4 \\
4: & \quad 4x_1 + 2x_2 - 2x_3 + x_4 + x_5 \leq 3 \\
2: & \quad 2x_1 + 4x_2 - 4x_3 - 2x_4 + 5x_5 \leq 5 \\
3: & \quad 3x_1 + x_2 + 2x_3 - x_4 - 2x_5 \leq 1 \\
   & \quad x_1, x_2, x_3, x_4, x_5 \geq 0
\end{align*} \]

Proposed Solution

\[ \begin{align*}
x_1^* &= 0 \quad x_2^* = \frac{4}{3} \quad x_3^* = \frac{2}{3} \quad x_4^* = \frac{5}{3} \quad x_5^* = 0
\end{align*} \]

Check

\[ \begin{align*}
x_1^* + 3x_2^* + 5x_3^* - 2x_4^* + 2x_5^* = 3\left(\frac{4}{3}\right) + 5\left(\frac{2}{3}\right) - 2\left(\frac{5}{3}\right) = 4 \quad \text{(No Slack)}
\end{align*} \]

\[ \begin{align*}
4x_1^* + 2x_2^* - 2x_3^* + x_4^* + x_5^* = 4\left(\frac{4}{3}\right) + 2\left(\frac{4}{3}\right) - 2\left(\frac{5}{3}\right) + 5\left(\frac{3}{3}\right) = 3 \quad \text{(No Slack)}
\end{align*} \]

Also

\[ \begin{align*}
2x_1^* + 4x_2^* + 4x_3^* - 2x_4^* + 5x_5^* = 2\left(\frac{10}{3}\right) + 4\left(\frac{4}{3}\right) + 4\left(\frac{2}{3}\right) - 2\left(\frac{5}{3}\right) = 8 - \frac{10}{3} = \frac{14}{3} < 5
\end{align*} \]

Finally

\[ \begin{align*}
3x_1^* + x_2^* + 2x_3^* - x_4^* - 2x_5^* = 3\left(\frac{10}{3}\right) + 4\left(\frac{2}{3}\right) - \frac{5}{3} = 1 \quad \text{(No Slack)}
\end{align*} \]
Let \( y_1, y_2, y_3, \) and \( y_4 \) be the dual variables corresponding to the four primal constraints. Let \( y_1^*, y_2^*, y_3^*, \) and \( y_4^* \) be the values of these variables. From the complementary slackness condition, we have
\[
y_3^* = 0
\]
Also
\[
\begin{align*}
3y_1^* + 2y_2^* + 4y_3^* + y_4^* &= 6 \\
5y_1^* + (-27)y_2^* + 4y_3^* + 2y_4^* &= 5 \\
-2y_1^* + y_2^* - 2y_3^* - y_4^* &= -2
\end{align*}
\]
Substituting \( y_3^* = 0 \) in the last three equations, we get
\[
\begin{align*}
3y_1^* + 2y_2^* + y_4^* &= 6 \\
5y_1^* - 2y_2^* + 2y_4^* &= 5 \\
-2y_1^* + y_2^* - y_4^* &= -2
\end{align*}
\]
Solving these equations gives \( y^* = \left( \frac{1}{10} \right) \).

Now \( y^* = \left( \frac{1}{10} \right) \) violates the fourth (4th) constraint \( 2y_1 + 4y_2 + 3y_3 + 2y_4 \geq 7 \) in the dual problem, so \( x^* = (0, 4/3, 2/3, 5/3, 0) \) is not optimal.
Consider

\[ z = \text{Max } 6x_1 + x_2 - x_3 - x_4 \]

s.t.
\[ x_1 + 2x_2 + x_3 + x_4 \leq 5 \]
\[ 3x_1 + x_2 - x_3 \leq 8 \]
\[ x_2 + x_3 + x_4 = 1 \]
\[ x_1, x_4 \geq 0 \]

Let \( x_5, x_6, x_7 \) be the slack variables in the three functional constraints of LP.

Note \( x_7 = 0 \) (since the last functional constraint is an equality constraint).

The dual is

\[ w = \text{Min } 5y_1 + 8y_2 + y_3 \]

s.t.
\[ y_1 + 3y_2 = 6 \]
\[ 2y_1 + y_2 + y_3 = 1 \]
\[ y_1 - y_2 + y_3 = 7 \]
\[ y_1 + y_3 = 7 \]
\[ y_1, y_2 \geq 0 \]
\[ y_3 \text{ FREE} \]

Let \( y_4, y_5, y_6, y_7 \) be the surplus variables in the functional constraints of dual LP.

Note \( y_4 = 0, y_5 = 0 \) (since the 1st two constraints are equality constraints).
We have
\[ x^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*, x_7^*) = (3, -1, 0, 1, 2, 2, 0, 0). \]

The complementary variables are
\[
\begin{pmatrix}
  x_1^* \\
  x_2^* \\
  x_3^* \\
  x_4^* \\
  x_5^* \\
  x_6^* \\
  x_7^*
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  y_4 \\
  y_5 \\
  y_6 \\
  y_7 \\
  y_1 \\
  y_2 \\
  y_3
\end{pmatrix}
\]

From the complementary slackness conditions, we have
\[ y_4 = 0, \quad y_5 = 0, \quad y_7 = 0, \quad y_1 = 0. \]

Since \( y_7 = 0 \) we have \( y_1 + y_3 = -1 \) and so \( y_3 = -1 \).

Also \( 2y_1 + y_2 + y_3 = 1 \)
\[ y_2 = 2 \]

Finally, \( y_1 - y_2 + y_3 - y_6 = -1 \)
and so \( y_6 = y_1 - y_2 + y_3 - 1 \)
\[ = 0 - 2 - 1 - 1 \]
\[ = -4 \leq 0 \]
(3) Let \( u, v, w, \) and \( z \) be the dual variables corresponding to the constraints \( Ax \geq b, Ax \leq b, u \geq x \geq d, \) and \( x < u \) respectively.

The dual is

\[
\begin{align*}
U &= \text{Min} \quad b_T u + b_T v + l_T w + u_T z \\
\text{s.t} \quad A u + A_T v + w + z &= c \quad (u \leq 0) \quad (v \geq 0) \\
&\quad u \in \mathbb{R}^m, v \in \mathbb{R}^m, \\
&\quad w \leq 0, \quad w \in \mathbb{R}^n, \\
&\quad z \geq 0, \quad z \in \mathbb{R}^n
\end{align*}
\]

(5) Feasible solution \( u \)

\[
\begin{align*}
u &= 0, \quad v = 0, \quad \text{and} \quad w_j &= c_j, \quad y \quad c_j \leq 0 \\
&\quad y \quad c_j > 0
\end{align*}
\]

If the primal problem is also feasible, then by duality theory both problems are optimal with equal optimal objective values.

(c) Complementary Slackness conditions all
\[ u_i (Ax - b_i) = 0 \quad i = 1, 2, \ldots, m \]
\[ v_j (b_j - Ax_j) = 0 \quad j = 1, 2, \ldots, n \]
\[ w_k (x_k - x_k) = 0 \quad k = 1, 2, \ldots, n \]
\[ z_l (u_l - x_l) = 0 \quad l = 1, 2, \ldots, n \]

\((2m + 2n)\) conditions in all

3(a) Dual is

\[ \text{Max } b_1^T u + b_2^T w + x_1^T v + x_2^T w \]

s.t.

\[ AT u + AX w + w + z = c \]

\[ u \geq 0 \]
\[ w \leq 0 \]
\[ v \geq 0 \]
\[ 2 \leq 0 \]

1st system solvable \(\rightarrow\) 2nd system is unsolvable

(a) Given \(Ax = 0\), \(x > 0\). Consider any \(y\) satisfying \(AT y > 0\). We have

\[ X^T AT y = y^T (Ax) = 0 \] since \(Ax = 0\).

On the other hand \(X^T (AT y) \geq 0\) since \(x \geq 0\) and \(AT y > 0\) and the only way it will be 0 is if \(AT y = 0\).

So, the 2nd system \(AT y > 0\) and \(AT y = 0\) is unsolvable.
1. System \( \Rightarrow \) 2. System

4.5. Since the 1st system is unsolvable, there is no \( x \) satisfying \( Ax = 0 \) and \( x \geq e \) (using the hint)

Consider the LP

\[
\begin{align*}
Z = \text{Max } & \quad D^T x \\
\text{s.t.} & \quad Ax = 0 \\
& \quad x \geq e
\end{align*}
\]

This LP is infeasible. By duality theory, the dual LP

\[
\begin{align*}
W = \text{Min } & \quad D^T y + e^T z \\
\text{s.t.} & \quad A^T y + z = 0 \\
& \quad z \leq 0
\end{align*}
\]

is either infeasible or unbounded.

But \( y = 0 \) and \( z = 0 \) is feasible in the LP.

So the dual is unbounded, i.e., there exists some \( d \) satisfying 

\[ e^T d < 0 \quad \text{and} \quad A^T y + d = 0 \]

(Why?)

Since \( e^T d < 0 \) and \( d \leq 0 \), we must have \( d = 0 \).

\[ \Rightarrow \text{we have} \quad A^T y = -d > 0 \quad \text{and} \quad A^T y = 0 \]

i.e., the 2nd system is solvable.
Consider the LP in standard form:

\[ z = \text{Max } c^T x \]
\[ s + A x \leq b \]
\[ x \geq 0 \]

The dual LP is:

\[ w = \text{Min } b^T y \]
\[ s + A^T y \geq c \]
\[ y \geq 0 \]

We can write the dual as a maximization problem as follows:

\[ w = \text{Max } -b^T y \]
\[ s + -A^T y \leq -c \]
\[ y \geq 0 \]

Now \((i)\) and \((ii)\) are equivalent if and only if

\[ c = -b \text{ and } A = -A^T \]

Note that \(A = -A^T\) means that \(A\) has to be a skew-symmetric matrix. The diagonal entries of a skew-symmetric matrix have to be zero. (Why?)
(c) For example consider
\[ z = \text{Max } 2x_1 + 3x_2 \]
subject to
\[ s.t. \quad 2x_2 \leq -2 \]
\[ -2x_1 \leq -3 \]
\[ x_1, x_2 \geq 0 \]

(d) (i) A self-dual LP cannot be unbounded.
(ii) It can be infeasible.
(iii) It can be optimal, i.e., have an optimal solution.

(See Table 5.1 in CIVATAC for the reasons behind these 3 statements.)

(6) (a) Introduce an additional variable \( t \) and rewrite the original LP as
\[ z = \text{Min } t \]
\[ s.t. \quad x_1, x_2, t \]
\[ \text{Max } \left\{ 2x_1 - x_2 : -3x_1 + 2x_2 + x_2 \right\} \leq t \]
\[ x_1 + x_2 + x_3 = 1 \]
\[ x_1, x_2, x_3 \geq 0 \]

OR
(b) The dual is

\[ w = \max \ y_3 \]

\[ s.t. \]
\[ -2y_1 + 3y_2 + y_3 \leq 0 \]
\[ y_1 - 2y_2 + y_3 \leq 0 \]
\[ -y_2 + y_3 \leq 0 \]
\[ y_1 + y_2 = 1 \]
\[ y_1, y_2 \geq 0 \]

(c) \[ w = \max \ \min \ \left[ 2y_1 - 3y_2 \right] \]
\[ y_1, y_2 \]

This is the row player's problem. He is trying to maximize the minimum payoff he is likely to win in the game.
(a) The dual LP is:

\[ Z = \text{Max } 3y_2 \]
\[ \text{s.t. } -2y_1 + 3y_2 + y_3 \leq 0 \]
\[ y_1 - 2y_2 + y_3 \leq 0 \]
\[ y_2 + y_3 \leq 0 \]
\[ y_1 + y_2 = 1 \]
\[ y_1, y_2 \geq 0 \]

Use \( y_1 + y_2 = 1 \) to eliminate \( y_2 \) as \( y_2 = (1-y_1) \), Note \( y_2 \geq 0 \)
\[ \Rightarrow y_1 \leq 1 \]

The LP can be rewritten as:

\[ Z = \text{Max } 3y_2 \]
\[ \text{s.t. } -5y_1 + y_3 \leq -3 \]
\[ 3y_1 + y_3 \leq 2 \]
\[ y_1 + y_3 \leq 1 \]
\[ 0 \leq y_1 \leq 1 \]

Optimal solution:

\[ y_1^* = \frac{5}{8}, y_3^* = 1/8 \]
\[ y_2^* = (1-y_1^*) \]
\[ = 3/8 \]

The optimal dual solution:

\[ y^* = (y_1^*, y_2^*) = \left( \frac{5}{8}, \frac{3}{8} \right) \]
(c) From the complementary slackness conditions, we have

\[-2x_1 + x_2 + t = 0\]
\[3x_1 - 2x_2 - x_3 + t = 0\]
\[x_3 = 0\]

This gives \(x^* = (x_1^*, x_2^*, x_3^*) = \left(\frac{3}{8}, \frac{5}{8}, 0\right)\)

Also \(t^* = \frac{1}{8} \implies \text{the payoff}\)

(\(t^*\) is the value of the payoff)

(f) Note that \(y_3\) in the DVM problem is the value of the payoff.

We have

\[t^* = 2y_1^* - 3y_2^*\]
\[t^* = -(y_1^* + 2y_2^*)\]
\[t^* < \frac{1}{12}\]

Suppose the column player played his three strategies with probabilities \(x_1^*, x_2^*, \text{ and } x_3^*\)

We have

\[t^* \left[x_1^* + x_2^* + x_3^*\right] < x_1^* \left[2y_1^* - 3y_2^*\right] + x_2^* \left(-y_1^* + 2y_2^*\right) + x_3^* \left(\frac{1}{12}\right)\]
\[ t^* < x_1^* \left( 2y_1^* - 3y_2^* \right) + x_2^* \left( -y_1^* + 2y_2^* \right) + x_3^* (12^*) \]

\[
= \begin{bmatrix} y_1^* & y_2^* \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix}
\]

\[
\text{The money that exchanges hands when row player plays his strategies with probabilities } y_1^* \text{ and } y_2^*, \text{ and column player plays his strategies with probabilities } x_1^*, x_2^*, x_3^*. \]

So, if the column player plays his best strategy with a nonzero probability, he risks losing more than the optimal payoff \( t^* \).

So, he should choose \( x_2^* = 0 \).

The other complementary slackness conditions can be interpreted similarly.