INSTRUCTIONS

Due in class on Thursday, March 22, 2007. No late homeworks will be accepted without prior instructor approval. All problems are from Chvátal unless otherwise specified. Please read Chapters 7 and 10 in Chvátal before beginning the assignment.

1. Problem 7.1, Page 116. Solve Problem 2.1(a) only! Do only the first two iterations by hand. You can complete the rest of iterations using the revised simplex code that I have posted on the course webpage.

2. Consider the following LP

\[
\begin{align*}
\text{max} & \quad Z = -x_1 - x_2 \\
\text{s.t.} & \quad -2x_1 - x_2 \leq 4 \\
& \quad -2x_1 + 4x_2 \leq -8 \\
& \quad -x_1 + 3x_2 \leq -7 \\
& \quad x_1, x_2 \geq 0 
\end{align*}
\]

(a) Solve the LP using the two phase simplex method.

(b) Solve the LP using the dual simplex method, i.e., by applying the simplex method on dual dictionaries as discussed in class.

(c) Which method is more efficient with regard to the number of iterations? For the two phase method, you should count the number of iterations in both phases of the simplex method.

(d) Solve the LP using the revised dual simplex method (see Box 10.1 on page 155 of Chvátal). Do only the first two iterations by hand. You can complete the rest of the iterations using the dual simplex code that I have posted on the course webpage. Verify that your results in the various iterations are identical to those in part (b).

3. Problem 10.2, Page 167. In each case, use sensitivity analysis to reoptimize the new LP.
4. Consider the optimal dictionary
\[
\begin{align*}
x_1 &= 2 + x_4 - \frac{1}{2}x_6 - \frac{1}{5}x_7 + x_8 \\
x_2 &= 3 - 2x_4 - x_5 + x_6 - \frac{1}{2}x_8 \\
x_3 &= 1 + x_4 + 2x_5 - 5x_6 + \frac{3}{10}x_7 - 2x_8 \\
z &= \theta - x_4 - 2x_6 - \frac{1}{10}x_7 - 2x_8.
\end{align*}
\]
of an LP (maximization problem and all \(\leq\) constraints), where \(x_6, x_7,\) and \(x_8\) are the slack variables in the problem.

(a) Find the optimal objective value \(\theta\).
(b) Would the solution be altered if a new activity \(x_9\) with coefficients \((6, 0, -3)^T\) in the constraints and a cost coefficient 7 were added to the problem?
(c) How large can \(b_1\) (rhs of the first constraint) be made without violating feasibility?
(d) Suppose we add the constraint \(2x_1 - x_2 + 2x_3 \leq 2\) to the problem. Is the earlier solution still optimal? If not, find a new optimal solution.

5. The following example illustrates an important application of linear programming together with the primal and dual simplex methods in solving integer programs. Integer programs arise naturally in real life applications, but are normally very difficult to solve! Consider the following integer program
\[
\begin{align*}
\text{max} & \quad Z = x_2 \\
\text{s.t.} & \quad 3x_1 + 2x_2 \leq 6 \\
& \quad -3x_1 + 2x_2 \leq 0 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0 \\
& \quad x_1, x_2 \text{ integer}
\end{align*}
\]
(1)

(a) Sketch the feasible region of (1). To do this, sketch the feasible region of the linear program (LP) obtained by dropping the integrality restrictions on \(x_1\) and \(x_2\), and then label all the integer points \((x_1, x_2 \text{ integer})\) within the LP feasible region. How many integer points do you find? What is the optimal solution to (1)?
(b) We will develop a step by step technique due to Ralph Gomory in the 1950s for solving (1) as follows: First solve an LP relaxation of (1). This LP is obtained simply by dropping the integrality restrictions on \(x_1\) and \(x_2\) in (1). Let \(x_3\) and \(x_4\) be the slack variables corresponding to the first two functional constraints in the LP. Use the primal simplex method with dictionaries to solve the LP relaxation to optimality. You will need two iterations and your optimal dictionary will be
\[
\begin{align*}
x_1 &= 1 - \frac{1}{6}x_3 + \frac{1}{6}x_4 \\
x_2 &= \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4 \\
Z &= \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4.
\end{align*}
\]
(2)
(c) The solution to the LP relaxation is $x^* = (1, \frac{3}{2})$, where $x^*_2$ is not integer. However, we know from part (a) that there is no point with $x_2 > 1$ in the feasible region of the integer program. We add the constraint $x_2 \leq 1$ to the optimal dictionary (2) of the linear program, which preserves all the integer solutions but cuts off the point $x^*$. This constraint is equivalent to $\frac{1}{4}x_3 + \frac{1}{4}x_4 \geq \frac{1}{2}$? Why? Adding this constraint to (2) gives the following dictionary

$$
\begin{align*}
  x_1 &= 1 - \frac{1}{6}x_3 + \frac{1}{6}x_4 \\
  x_2 &= \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4 \\
  x_5 &= -\frac{1}{2} + \frac{1}{4}x_3 + \frac{1}{4}x_4 \\
  Z &= \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4,
\end{align*}
$$

(3)

where $x_5$ is the surplus variable associated with the new constraint. Note that (3) is no longer a primal feasible dictionary but it is dual feasible! Why? This means we can apply the primal simplex method to the dual problem. Construct the dictionary for the dual problem using the correspondence between equations (10.4) and (10.6) on page 150 of Chvátal. You only need one iteration of the simplex method and you are optimal! The optimal dictionary for the new problem is

$$
\begin{align*}
  x_1 &= \frac{2}{3} + \frac{1}{3}x_4 - \frac{2}{3}x_5 \\
  x_2 &= 1 + 0x_4 - x_5 \\
  x_3 &= 2 - x_4 + 4x_5 \\
  Z &= 1 + 0x_4 - x_5,
\end{align*}
$$

(4)

which is again not optimal for (1) since $x^*_1$ is not integer.

(d) We know from part (a) that $x_1 \geq x_2$ for all the feasible points in (1). So, we will add this constraint to (4) as

$$
\begin{align*}
  x_6 &= -\frac{2}{3} + \frac{2}{3}x_4 + \frac{2}{3}x_5,
\end{align*}
$$

where $x_6$ is the surplus variable associated with the new constraint. This constraint preserves all the integer solutions but cuts off the point $x^* = (\frac{2}{3}, 1)$. Note that our new dictionary is again primal infeasible but dual feasible! Apply one more iteration of the simplex method to the dictionary associated with the dual
problem. This will give you the dictionary

\[
\begin{align*}
x_1 &= 1 - x_5 + \frac{1}{2}x_6 \\
x_2 &= 1 - x_5 + 0x_6 \\
x_3 &= 1 + 5x_5 - \frac{3}{2}x_6 \\
x_4 &= 1 - x_5 + \frac{3}{2}x_6 \\
Z &= 1 - x_5 + 0x_6,
\end{align*}
\] (5)

which is optimal for (1) since \( x_1 \) and \( x_2 \) are finally integer! This gives the optimal solution \( x^* = (1, 1) \) for (1).

\( (e) \) Plot the two cutting planes \( x_2 \leq 1 \) and \( x_1 \geq x_2 \) on the plot from part (a). Also, trace the path taken by the algorithm to find the optimal solution to (1) here.