MA / IE / OR 505

SOLUTIONS TO SELECTED REVIEW PROBLEMS

(PREPARED BY KARTIK)

NOTE:—
I AM ENCLOSING THE SOLUTIONS TO 1, 2, 5, 7, and 10.

DO THE REST ON YOUR OWN.

(1) (i) $s = 2,\ t = 1$
LP has finite optimal solution
What happens to the dual problem?

(ii) $s = 1,\ t = 1$
LP has multiple optimal solutions.

(iii) $s = 0,\ t = 1$
LP is unbounded.

(iv) $s = 1,\ t = -1$ LP is infeasible

(2)

(a) $\beta \geq 0$

(b) $\beta < 0$ and $\alpha \geq 0$
This gives
$$\underbrace{x_2 + \alpha x_4 + x_5 + 3x_7}_{\geq 0} = \beta \ (< 0)$$
(since $x_2, \alpha, x_4, x_5, x_7 \geq 0$) A CONTRADICTION

(c) $\beta \geq 0$ and ($\delta > 0$ OR $\gamma > 0$ OR $\xi < 0$)

(d) $\delta > \text{Max}\{\gamma, -\Phi\}$ for $\gamma > 0$ and $\Phi < 0$

So that x_4 enters the basis and $\beta \geq 0$ and $\alpha \leq 0$

Since, $\beta \geq 0$ and $\alpha \leq 0$ no basic variable will leave the basis and the LP will be unbounded.

(e) $\gamma > \text{Max}\{\delta, -\Phi\}$ for $\delta > 0$ and $\Phi < 0$

This will ensure that x_6 enters the basis.

We then have

$x_3 = 2 - \eta x_6 \geq 0$

$x_1 = 3 - 2x_6 \geq 0$

So, if $2/\eta < 3/2$ then x_3 leaves the basis

(f) $-\Phi > \text{Max}\{\gamma, \delta\}$ for $\gamma, \delta > 0$ so x_7 enters the basis

But we have a degenerate basic feasible solution and so $\beta = 0$.

In this case, x_2 leaves the basis.

However, the current solution remains the same.

(7) (a) Choosing $x_j = 0$, $j = 1, 2, \ldots, n$
and setting x_0 to an arbitrarily large value (anything greater than or equal the negative of the most +ve rhs coefficient) gives a feasible solution to the auxiliary problem

(b) DUAL IS

$$W = \text{MAX } b^T y \qquad \text{OR} \qquad \text{Max } -b^T y$$

$$\text{st } \quad A^T y \le 0 \qquad\qquad \text{st } \quad A^T y \ge 0$$

$$-e^T y \le 1 \qquad\qquad e^T y \le 1$$

$$y \le 0 \qquad\qquad y \ge 0$$

(c) The dual always possesses a feasible solution, since $y = 0$ is a feasible solution in this problem

(d) The complementary slackness conditions are

$$y_i \left(b_i - \sum_{j=1}^{n} a_{ij} x_j + x_0 \right) = 0, \quad i = 1, 2, \ldots, m$$

$$x_j (A^T y)_j = 0, \quad j = 1, 2, \ldots, n$$

$$x_0 (1 - e^T y) = 0$$

There are $(m+n+1)$ conditions

1(c) Since the primal and dual are both feasible, the two LPs will be optimal and their optimal objective values will be the same. In particular, the auxiliary problem cannot be unbounded.

(9)

(a) (1st system is solvable) $\Longrightarrow$ (2nd system is unsolvable)

Consider any y satisfying

$A^T y \geq 0$ and $y \geq 0$.

We have
$$X^T(A^T y) = (Ax)^T y \geq 0 \quad \left(\text{since } x \geq 0 \text{ and } A^T y \geq 0\right)$$

$\therefore (Ax)^T y \geq 0$

But $Ax < 0$ and $y \geq 0$ and we must have $y = 0$ to ensure $(Ax)^T y = (Ax)^T 0 = 0 \geq 0$.

This implies that the 2nd system is unsolvable.

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(b) $\left(\begin{array}{l}\text{1st system} \\ \text{is unsolvable}\end{array}\right) \Rightarrow \left(\begin{array}{l}\text{2nd system is} \\ \text{solvable}\end{array}\right)$

There is no $x \in \mathbb{R}^n$ satisfying $Ax < 0$ and $x \geq 0$

i.e there is no $x \in \mathbb{R}^n$ satisfying $Ax \leq -e$ and $x \geq 0$ (USING THE HINT)

Consider the LP

$$\begin{array}{ll} \text{Max} & 0^T x \\ \text{s.t} & Ax \leq -e \\ & x \geq 0 \end{array}$$

(This LP is infeasible)

The dual LP is

$$\begin{array}{ll} \text{MIN} & -e^T y \\ \text{s.t} & A^T y \geq 0 \\ & y \geq 0 \end{array}$$

(By duality theory, this LP is either infeasible or unbounded)

However, $y = 0$ is feasible in this LP. So the dual LP is unbounded.

This implies that there is some $d \in \mathbb{R}^m$ s.t

$-e^T d < 0$, $A^T d \geq 0$ and $d \geq 0$ (why?)

i.e $e^T d > 0$, $A^T d \geq 0$ and $d \geq 0$

This in turn implies that the 2nd system is solvable since such a d satisfies

$$A^T d \geq 0, \quad d \geq 0 \quad \text{and } d \neq 0$$

$$\left(\text{since } \sum_{i=1}^{M} d_i > 0\right)$$

(10) We have

$$x_B = A_B^{-1} b - A_B^{-1} A_N x_N$$

$$z = c_B^T A_B^{-1} b + (c_N - A_N^T y)^T x_N$$

where $A_B^T y = c_B$

In our case,

$$A_B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \qquad A_B^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

Now

$$\begin{bmatrix} \alpha \\ 3/5 \end{bmatrix} = A_B^{-1} \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{a_1} = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 3/5 \end{bmatrix}$$

$$\therefore \boxed{\alpha = 1/5}$$

(b) We have

$$z = c_1 x_1 + c_2 x_2 + c_3 x_3 = 0 - \frac{7}{10} x_1 - \frac{3}{5} x_4 - \frac{4}{5} x_5$$

Substituting the values of x_2 and x_3 from the optimal dictionary we have

$$c_1 x_1 + c_2\left(1 - \frac{1}{5}x_1 - \frac{3}{5}x_4 + \frac{1}{5}x_5\right) + c_3\left(3 - \frac{3}{5}x_1 + \frac{1}{5}x_4 - \frac{2}{5}x_5\right) = 0 - \frac{7}{10}x_1 - \frac{3}{5}x_4 - \frac{4}{5}x_5$$

This gives

$$c_2 + 3c_3 = 0$$

$$\left(c_1 - \frac{1}{5}c_2 - \frac{3}{5}c_3\right) = \frac{-7}{10}$$

$$\left(-\frac{3}{5}c_2 + \frac{1}{5}c_3\right) = -3/5$$

$$\left(\frac{1}{5}c_2 - \frac{2}{5}c_3\right) = -4/5$$

$\Rightarrow$ This gives $c_2 = 2$, $c_3 = 3$ and $c_1 = 3/2$

(c) We have

$$x_B = A_B^{-1} b \quad \therefore \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} b \\ 2b \end{bmatrix} = \frac{1}{5}\begin{bmatrix} b \\ 3b \end{bmatrix} \quad \therefore \boxed{b = 5}$$

(d) Solve $A_B^T y = c_B$

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow y_1 = 3/5 \text{ and } y_2 = 4/5$$

$$\therefore \boxed{y = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}}$$

(e) OPTIMAL OBJ VALUE

$$0 = c_2 + 3c_3 = 2 + 3(3) = 11$$