MA 5550

Solutions to Selected Review Problems

Note: I am enclosing the solutions to 1, 2, 3, 4, and 10.

Do the rest on your own.

1. (i) \( s = 2 \), \( t = 1 \)
   LP has finite optimal solution. What happens to the dual problem?
   
   (ii) \( s = 1 \), \( t = 1 \)
   LP has multiple optimal solutions.
   
   (iii) \( s = 0 \), \( t = 1 \)
   LP is unbounded.
   
   (iv) \( s = 1 \), \( t = -1 \) LP is infeasible.

2. (a) \( \beta > 0 \)
   (b) \( \beta < 0 \) and \( x > 0 \)
   This means:
   \[ x_2 + 2x_4 + x_5 + 3x_7 = \beta < 0 \]
   \[
   > 0
   \]
   (since \( x_2, x_4, x_5, x_7 > 0 \) A contradiction)
   
   (c) \( \beta > 0 \) and \( (\delta > 0 \text{ or } \gamma > 0 \text{ or } f < 0) \)
(d) $\mathcal{E} > \text{Max } \sum x_i - \mathcal{E} \text{ for } \mathcal{E} > 0$
and $\mathcal{E} < 0$

so that $x_4$ enters the basis
and $B > 0$ and $d < 0$

Since $B > 0$ and $d < 0$ no basic variable will leave the basis and the LP will be unbounded.

(c) $\mathcal{E} > \text{Max } \sum x_i - \mathcal{E} \text{ for } \mathcal{E} > 0$
and $\mathcal{E} < 0$

This will ensure that $x_3$ enters the basis.

We then have
$x_3 = 2 - \eta x_6 > 0$
$S_0, \eta \frac{2}{\eta} < 3/2$
$x_7 = 3 - 2x_6 > 0$ then $x_3$ leaves the basis

(b) $-\mathcal{E} > \text{Max } \sum x_i - \mathcal{E} \text{ for } \mathcal{E} > 0$

But we have a degenerate basic feasible solution
and $l_0 = 0$.

In this case, $x_2$ leaves the basis.

However, the current solution remains the same.
(a) Choosing \( x_j = 0 \), \( j = 1, 2, \ldots, n \) and setting \( x_0 \) to an arbitrarily large value (anything greater than or equal to the negative of the most negative coefficient) gives a feasible solution to the auxiliary problem.

(b) Dual is

\[
\begin{align*}
W &= \text{Max } b^T y \
\text{s.t. } A_y \leq 0 & \quad \text{s.t. } A_y^T y \geq 0 \\
-\epsilon^T y & \leq 1 \quad \epsilon^T y \leq 1 \\
& \quad y \leq 0, \quad y \geq 0
\end{align*}
\]

(c) The dual always possesses a feasible solution since \( y = 0 \) is a feasible solution in this problem.

(d) The complementary slackness conditions are

\[
\begin{align*}
y_i \left( k_i - \sum_{j=1}^{n} a_{ij} x_j + x_0 \right) &= 0, \quad i = 1, 2, \ldots, m \\
x_j \left( A_y^T \right)_j &= 0, \quad j = 1, 2, \ldots, n \\
x_0 \left( 1 - \epsilon^T y \right) &= 0 \quad \text{The above} \\
& \quad \text{(m+n+1) conditions}
\end{align*}
\]
(c) Since the primal and dual are both feasible, the two LPs will be optimal and their optimal objective values will be the same. In particular, the auxiliary problem cannot be unbounded.

(9) (1st system
(a) is solvable)

\[ \implies (2\text{nd system is unsolvable}) \]

Consider any \( y \) satisfying

\[ A^T y \geq 0 \text{ and } y \geq 0 \]

We have

\[ X^T (A^T y) = (Ax)^T y \geq 0 \quad (\text{since } x \geq 0 \text{ and } A^T y \geq 0) \]

\[ (Ax)^T y \geq 0 \]

But \( Ax \leq 0 \) and \( y \geq 0 \) and we must have

\[ y = 0 \text{ to ensure } (Ax)^T y = (Ax)^T 0 = 0 \geq 0 \]

This implies that the 2nd system is unsolvable.
5. (1st system is unsolvable) =) (2nd system is solvable)

There is no \( x \in \mathbb{R}^n \) satisfying \( Ax \leq 0 \) and \( x \geq 0 \).

I.e., there is no \( x \in \mathbb{R}^n \) satisfying \( Ax \leq -\epsilon \) and \( x \geq 0 \) (using the hint).

Consider the LP:

\[
\begin{align*}
\text{Max} & \quad 0^T x \\
\text{s.t} & \quad Ax \leq -\epsilon \\
& \quad x \geq 0
\end{align*}
\]

This LP is infeasible (This LP is infeasible).

The dual LP is:

\[
\begin{align*}
\text{Min} & \quad -\epsilon^T y \\
\text{s.t} & \quad A^T y \geq 0 \\
& \quad y \geq 0
\end{align*}
\]

However, \( y = 0 \) is feasible in this LP.

So the dual LP is unbounded.

This implies that there is some \( \delta \in \mathbb{R}^m \) s.t.

\[
\begin{align*}
-\epsilon^T d & \leq 0, \\
\epsilon^T d & \geq 0, \quad \text{and } d \geq 0
\end{align*}
\]

(why?)

\[
\begin{align*}
\epsilon^T d & \geq 0, \\
\epsilon^T d & \geq 0, \quad \text{and } d \geq 0
\end{align*}
\]
This in turn implies that the 2nd system is solvable.

Since such a $d$ satisfies

$$A^Td = 0, \quad d > 0 \quad \text{and} \quad d \neq 0$$

($\text{since } \sum_{i=1}^{N} d_i > 0$).

(10) We have

$$x_B = A_B^{-1}b - A_B^{-1}A_Nx_N$$

$$z = c_B^T A_B^{-1} b + (c_N - A_N y) ^T x_N$$

Where $A_B^Ty = c_B$.

In our case,

$$A_B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \quad A_B^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

Now

$$\begin{bmatrix} x \\ 3/5 \end{bmatrix} = A_B^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$a_1 = \begin{bmatrix} 1/5 \\ 3/5 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 1/5 \\ 3/5 \end{bmatrix}$$
We have
\[ 2 = c_1 x_1 + c_2 x_2 + c_3 x_3 = \frac{-1}{10} x_1 - \frac{2}{5} x_4 - \frac{4}{5} x_5 \]

Substituting the values of \( x_2 \) and \( x_3 \) from the optimal dictionary,
we have
\[ c_1 x_1 + c_2 \left(1 - \frac{1}{5} x_1 - \frac{2}{5} x_4 + \frac{1}{5} x_5\right) \]
\[ + c_3 \left(3 - \frac{3}{5} x_1 + \frac{1}{5} x_4 - \frac{3}{5} x_5\right) = \frac{-2}{10} x_1 - \frac{2}{5} x_4 - \frac{4}{5} x_5 \]

This gives
\[ c_2 + 3 c_3 = 0 \]
\[ \left(\frac{4}{5} c_2 - \frac{3}{5} c_3\right) = -\frac{2}{10} \quad \Rightarrow \quad \text{This} \]
\[ \left(-\frac{2}{5} c_2 + \frac{1}{5} c_3\right) = -\frac{2}{5} \quad \text{gives} \]
\[ \left(\frac{1}{5} c_2 - \frac{2}{5} c_3\right) = -\frac{4}{5} \quad c_2 = 2, \quad c_3 = 3 \]
and \( c_1 = 3/2 \).

We have
\[ x_0 = A_0^{-1} b \quad \Rightarrow \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 & -1 \\ 2 & b \end{bmatrix} \quad \begin{bmatrix} b \\ 3 b \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix} \quad \Rightarrow \quad b = 5 \]

Solve \( A_0^t y = c_0 \)
\[ \begin{bmatrix} 2 & 1 \\ 12 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \Rightarrow \quad y_1 = \frac{3}{5} \quad \text{and} \quad y_2 = \frac{4}{5} \]
\[ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \]

Optimal value \( Q = c_2 + 3 c_3 = 2 + 3(3) = 11 \)