

MA/IE/OR 505-001: Linear Programming  
Spring 2007  
Midterm Review  
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## INSTRUCTIONS

The questions on the midterm will be similar to these 10 review problems. Work out each problem in sufficient detail. If you have any questions, please send me an email. I will post the solutions to selected problems on the course webpage on Tuesday.

1. Consider the linear programming problem

$$\begin{array}{ll} \max & x_1 + x_2 \\ \text{s.t.} & sx_1 + x_2 \leq t \\ & x_1, x_2 \geq 0. \end{array}$$

- (a) Find some values of  $s$  and  $t$  such that this linear program has a) unique optimum solution, b) multiple optimal solutions, c) no feasible solution (LP is infeasible), and d) no optimal solution (LP is unbounded). Illustrate each case with a figure.
  - (b) Write down the dual LP for each of the four cases. What happens to the dual linear program in each of the four cases.
2. Consider the following dictionary associated with a linear program in standard form

$$\begin{array}{rcl} x_2 & = & \beta - \alpha x_4 - x_5 - 3x_7 \\ x_3 & = & 2 + 2x_4 - 2x_5 - \eta x_6 + x_7 \\ x_1 & = & \frac{3 + x_5 - 2x_6 - x_7}{z} \\ z & = & 0 + \delta x_4 - 3x_5 + \gamma x_6 - \phi x_7. \end{array}$$

The entries  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ ,  $\eta$ , and  $\phi$  in the dictionary are unknown parameters. For each one of the following statements, find some parameter values that will make the following statements true.

- (a) The simplex method can be applied using this as an initial dictionary.
- (b) The first row of the dictionary indicates that the problem is infeasible.
- (c) The corresponding basic solution is feasible, but we do not have an optimal solution.

- (d) The corresponding basic solution is feasible, and the first simplex iteration indicates that the problem is unbounded.
- (e) The corresponding basic solution is feasible;  $x_6$  is a candidate for entering the basis; and when  $x_6$  enters the basis,  $x_3$  leaves the basis.
- (f) The corresponding basic solution is feasible;  $x_7$  is a candidate for entering the basis; but if it does, the solution and the objective value remain unchanged.
3. Use a linear programming formulation to show that the following constraints

$$\begin{aligned} 4x_1 + x_2 &\leq 4 \\ 2x_1 - 3x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

imply  $x_1 + 2x_2 \leq 8$ .

4. Verify that  $x = (6, 0, 1, 0, 1, 0, 0)$  is an optimal solution to

$$\begin{aligned} \min \quad & -2x_1 + 13x_2 + 3x_3 - 2x_4 + 5x_5 + 5x_6 + 10x_7 \\ \text{s.t.} \quad & x_1 - x_2 + 4x_4 - x_5 + x_6 - 4x_7 = 5 \\ & x_1 + 7x_4 - 2x_5 + 3x_6 - 3x_7 \geq -1 \\ & 5x_2 + x_3 - x_4 + 2x_5 - x_6 - 2x_7 \leq 5 \\ & 3x_2 + x_3 + x_4 + x_5 + x_6 - x_7 = 2 \\ & x_i \geq 0, \quad i = 1, \dots, 7 \end{aligned}$$

using the supervisor principle.

5. Consider the linear programming problem

$$\begin{aligned} \min \quad & x_1 - x_2 \\ \text{s.t.} \quad & 2x_1 + 3x_2 - x_3 + x_4 \leq 0 \\ & 3x_1 + x_2 + 4x_3 - 2x_4 \geq 3 \\ & -x_1 - x_2 + 2x_3 + x_4 = 6 \\ & x_1 \leq 0 \\ & x_2, x_3 \geq 0. \end{aligned}$$

Write down the corresponding dual problem.

6. Consider the following problem

$$\begin{aligned} \min \quad & 2x_1 + 15x_2 + 5x_3 + 6x_4 \\ \text{s.t.} \quad & x_1 + 6x_2 + 3x_3 + x_4 \geq 2 \\ & -2x_1 + 5x_2 - 4x_3 + 3x_4 \leq -3 \\ & x_i \geq 0, \quad i = 1, \dots, 4. \end{aligned}$$

- (a) Give the dual linear program.
- (b) Solve the dual problem geometrically.
- (c) Use the dual optimal solution and the complementary slackness conditions to solve the primal problem.

7. Consider the so-called *auxiliary problem*

$$\begin{aligned} \min \quad & x_0 \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij}x_j - x_0 \leq b_i, \quad i = 1, \dots, m \\ & x_0 \geq 0 \\ & x_j \geq 0, \quad j = 1, \dots, n \end{aligned}$$

that we encountered during initialization (phase 1) of the standard simplex method.

- Show that the auxiliary problem always possesses a feasible solution.
  - Write down the dual to the auxiliary problem.
  - Show that the dual problem always possesses a feasible solution too.
  - What are the complementary slackness conditions for this primal-dual pair of LPs?
  - Using duality theory, what conclusions can you reach about the auxiliary problem and its dual? In particular, can the auxiliary problem be unbounded?
8. Consider the one constraint (knapsack) LP

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & a^T x \leq b \\ & x \geq 0 \end{aligned}$$

where  $b$  is a positive scalar;  $c, a \in \mathbb{R}^n$  with  $c_j, a_j > 0, j = 1, \dots, n$ . Suppose we also have  $\frac{c_1}{a_1} < \frac{c_2}{a_2} < \dots < \frac{c_n}{a_n}$ .

- Write the dual to this linear program.
- Construct the optimal solution to the dual problem.
- Use complementary slackness conditions to find the corresponding primal solution. Verify that the primal and dual objective values are the same.
- Verify your observations by solving the following knapsack LP

$$\begin{aligned} \max \quad & 3x_1 + 5x_2 \\ \text{s.t.} \quad & 4x_1 + 5x_2 \leq 10 \\ & x_1, x_2 \geq 0, \end{aligned}$$

using the revised simplex method.

9. Let  $A \in \mathbb{R}^{m \times n}$ ,  $x \in \mathbb{R}^n$ , and  $y \in \mathbb{R}^m$ . Derive the following theorem of the alternative

**Theorem 1** *The system  $Ax < 0, x \geq 0$  is solvable if and only if the system  $A^T y \geq 0, y \geq 0, y \neq 0$  is unsolvable.*

using LP duality (along the lines of the proof of Farkas' theorem in class). Note that the strict vector inequality  $y > 0$  implies that  $y_i > 0$  for all  $i$ . Proceed as follows:

- (a) Show that if the 1st system is solvable then the 2nd system is unsolvable.
- (b) Show that if the 1st system is unsolvable then the 2nd system is solvable. (**Hint:** Rewrite the 1st system as  $Ax \leq -e$ ,  $x \geq 0$  (where  $e$  is the  $m$  dimensional vector of all ones) since LPs deal with inequalities rather than strict inequalities!)

10. Consider the following problem

$$\begin{aligned} \max \quad & Z = c_1x_1 + c_2x_2 + c_3x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 \leq b \\ & 2x_1 + x_2 + 3x_3 \leq 2b \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Note that we have not yet assigned values to the coefficients  $c_1$ ,  $c_2$ , and  $c_3$  in the objective function, and the only specification for the right hand side of the functional constraints is that the second one ( $2b$ ) be twice as large as the first ( $b$ ). Now suppose Kartik has inserted his best estimate of the values of  $c_1$ ,  $c_2$ ,  $c_3$ , and  $b$  without informing you and then has run the simplex method. The optimal dictionary is given below (where  $x_4$  and  $x_5$  are the slack variables corresponding to the two functional constraints in the linear program), but you are unable to read the values of  $\alpha$  and  $\theta$  (owing to Kartik's bad handwriting :)

$$\begin{aligned} x_2 &= 1 - \alpha x_1 - \frac{3}{5}x_4 + \frac{1}{5}x_5 \\ x_3 &= 3 - \frac{3}{5}x_1 + \frac{1}{5}x_4 - \frac{2}{5}x_5 \\ \hline Z &= \theta - \frac{7}{10}x_1 - \frac{3}{5}x_4 - \frac{4}{5}x_5. \end{aligned}$$

Compute the following (in that order):

- (a) Find the value of  $\alpha$ .
- (b) Find the values of  $c_1$ ,  $c_2$ , and  $c_3$ .
- (c) Find the value of  $b$ .
- (d) Find the optimal solution  $y$  to the dual problem.
- (e) Calculate the optimal objective value  $\theta$ .