MA/OR/IE 505-001: Linear Programming
Midterm Exam
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SOLUTIONS TO THE MIDTERM EXAM
PREPARED BY KARTIK

INSTRUCTIONS

1. Please write your name and student number clearly on the front page of the exam.
2. This has to be your own work. Cheating on the exam is not tolerated, and will fetch you a zero for the test.
3. TIME LIMIT: 75 minutes
4. There are 5 pages and 3 questions on the exam. Each question appears on a different page. Read each question carefully.
5. The exam is worth 105 points of which 5 points are extra credit. The distribution of points is clearly indicated on the exam.
6. Solve each problem in sufficient detail in the space provided. Please use both sides of each page as needed.
7. Write clearly, including all the steps to the final solution. If I can't read it, you won't get credit.
8. Sources: Copy of Chvátal and one crib sheet.
9. You may use an electronic calculator on the exam.
10. A useful formula for the inverse of 2×2 invertible matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

1. [40 points] Consider a problem with 3 people to carry out 3 jobs. Each person is assigned to carry out exactly one job. There is a cost c_{ij} involved in training person i to do job j. The problem is to find a minimum cost assignment. Let $x_{ij} = 1$ if person i is assigned to job j and 0 otherwise. The assignment problem can be formulated as the following linear program (because the optimal solutions x_{ij} to the linear program will be either 0 or 1):

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij}x_{ij} & & \\
\text{s.t.} \quad & \sum_{j=1}^{3} x_{ij} &=& \ 1, \ i = 1,\ldots,3 \\
& \sum_{i=1}^{3} x_{ij} &=& \ 1, \ j = 1,\ldots,3 \\
& x_{ij} &\geq& \ 0, \ i = 1,\ldots,3, \ j = 1,\ldots,3.
\end{aligned} \tag{1}
$$

(a) [5 points] Find a feasible solution for the LP (1).

(b) [10 points] Let u_1, u_2, u_3 and v_1, v_2, v_3 be the dual variables corresponding to the first 3 and the last 3 equality constraints in (1) respectively. Formulate the dual LP.

(c) [10 points] Write down the complementary slackness conditions for the primal-dual pair of LPs. (Note: You must write down all the complementary slackness conditions explicitly to get credit on this problem).

(d) [5 points] Give a feasible solution for the dual LP.

(e) [5 points] Can the LP (1) and its dual be a) infeasible, b) unbounded, c) optimal? Give reasons to support your answer.

(f) [Extra credit: 5 points] Suppose some of the $c_{ij} < 0$, i.e., person i is extremely skilled at doing job j and needs no training. Find a feasible solution in the dual LP.

(i) The LP is

Max $c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{21}x_{21} + c_{22}x_{22} + c_{23}x_{23} + c_{31}x_{31} + c_{32}x_{32} + c_{33}x_{33}$

s.t.

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} x_{11} \\ \vdots \\ x_{33} \end{bmatrix}
=
\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}
\qquad
\begin{aligned}
& x_{ij} \geq 0 \\
& i = 1,2,3 \\
& j = 1,2,3
\end{aligned}
$$

2

A feasible solution is $x_{11} = 1$, $x_{22} = 1$, $x_{33} = 1$

and $x_{ij} = 0$, $i = 1,2,3$, $j = 1,2,3$ and $i \neq j$

1(b) Dual LP is

$$\text{Max} \quad \sum_{i=1}^{3} u_i + \sum_{j=1}^{3} v_j$$

$$\text{s.t.} \quad u_i + v_j \leq c_{ij}, \quad i = 1,2,3, \; j = 1,2,3$$

1(c) The complementary slackness conditions are

$$x_{ij}(u_i + v_j - c_{ij}) = 0, \quad i = 1,2,3, \; j = 1,2,3$$

There are 9 conditions in all.

1(d) Since $c_{ij} \geq 0$, $i = 1,2,3$, $j = 1,2,3$

a feasible solution for the dual problem is $u_1 = u_2 = u_3 = v_1 = v_2 = v_3 = 0$.

1(e) By duality theory, both problems are OPTIMAL and have equal optimal objective values. In particular, neither the primal nor the dual LP can be either infeasible or unbounded.

1(f) (EXTRA CREDIT)

$$\bar{u}_i = \min_{j=1,2,3} \{ c_{ij} - v_j \}, \quad i = 1,2,3$$

and

$$\bar{v}_j = \min_{i=1,2,3} \{ c_{ij} - \bar{u}_i \}, \quad j = 1,2,3$$

gives a feasible solution in the dual LP when some of the $c_{ij} < 0$

(why?)

2. [40 points] Consider the following problem

$$\begin{array}{lrcl} \max & 4x_1 + 3x_2 + x_3 + 2x_4 & & \\ \text{s.t.} & 4x_1 + 2x_2 + x_3 + x_4 & \leq & 5 \\ & 3x_1 + x_2 + 2x_3 + x_4 & \leq & 4 \\ & x_i & \geq & 0, \ i = 1, \ldots, 4. \end{array}$$

Let x_5 and x_6 denote the slack variables of the two functional inequalities in the LP (in that order!). After we apply the simplex method, we obtain the following optimal dictionary:

$$\begin{array}{rcl} x_2 & = & \alpha + ax_1 + bx_3 - x_5 + x_6 \\ x_4 & = & \beta + dx_1 + ex_3 + x_5 - 2x_6 \\ \hline Z & = & \theta + fx_1 + gx_3 - x_5 - x_6. \end{array}$$

(a) [25 points] Using the matrix representation of dictionaries (see equation (7.6) on page 99 of Chvátal), determine the values of $\alpha, \beta, \theta, a, b, d, e, f, g$.

(b) [5 points] Is the optimal solution nondegenerate? Give reasons for your answer.

(c) [5 points] Are there any multiple optimal solutions? Give reasons for your answer.

2 (a) We have

$$\begin{array}{rcl} x_B & = & A_B^{-1} b - A_B^{-1} A_N x_N \\ \hline z & = & c_B^T A_B^{-1} b + (c_N - A_N^T y)^T x_N \end{array}$$

where $A_B^T y = c_B$

In our case

$B = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $N = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 6 \end{bmatrix}$, $A_B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $A_N = \begin{bmatrix} 4 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{bmatrix}$

$c_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, and $c_N = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Also $A_B^{-1} = \frac{1}{1} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

We have

$x_B = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = A_B^{-1} b = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\therefore$ $\boxed{\alpha = 1 \\ \beta = 3}$

Also $\theta = c_B^T x_B = c_B^T A_B^{-1} b$

$= \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 9$ $\therefore$ $\boxed{\theta = 9}$

We also have

$$-A_B^{-1}A_N = -\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}4 & 1 & 1 & 0\\ 3 & 2 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}a & b & 1 & -1\\ d & e & -1 & 2\end{bmatrix}$$

$$\therefore \begin{bmatrix}a & b & 1 & -1\\ d & e & -1 & 2\end{bmatrix} = \begin{bmatrix}-1 & 1 & -1 & 1\\ -2 & -3 & 1 & -2\end{bmatrix}$$

$\therefore$

a = -1
b = 1
d = -2
e = -3

Also, $A_B^T y = c_B$

$$\therefore \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}3\\ 2\end{bmatrix} \quad \text{and } y = \begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix}$$

Finally $(c_N - A_N^T y) = \begin{bmatrix}4\\ 1\\ 0\\ 0\end{bmatrix} - \begin{bmatrix}4 & 3\\ 1 & 2\\ 1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}-3\\ -2\\ -1\\ -1\end{bmatrix}$

$\therefore$

f = -3
g = -3

(b) The optimal solution is nondegenerate since $X_B = \begin{bmatrix}1\\ 3\end{bmatrix} > \begin{bmatrix}0\\ 0\end{bmatrix}$, i.e, the basic variables are all positive

(c) There are no MULTIPLE optimal solutions since $S_N = \begin{bmatrix}-3\\ -2\\ -1\\ -1\end{bmatrix}$, i.e, the coefficients of all the nonbasic variables in the objective function row are all strictly negative.

3. [25 points] Let $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Derive the following theorem of the alternative

Theorem 1 *The system* $Ax < 0$ *is unsolvable if and only if the system* $A^T y = 0$, $y \geq 0$, $y \neq 0$ *is solvable.*

using LP duality (along the lines of the proof of Farkas' theorem in class). Note that the strict vector inequality $y > 0$ implies that $y_i > 0$ for all i. Also, $y \geq 0$ and $y \neq 0$ implies that $y_i \geq 0$ for all i and all the y_i cannot ~~all simultaneously~~ be 0. Proceed as follows:
SIMULTANEOUSLY

(a) [10 points] Show that if the 1st system is solvable then the 2nd system is unsolvable.

(b) [15 points] Show that if the 1st system is unsolvable then the 2nd system is solvable. (Hint: Rewrite the 1st system as $Ax \leq -e$, where e is the m dimensional vector of all ones, since LPs deal with inequalities rather than strict inequalities!)

3(a) (1st system is solvable) $\Rightarrow$ (2nd system is unsolvable)

Consider any $y \in \mathbb{R}^m$ and satisfying $A^T y = 0$ and $y \geq 0$

we have $x^T(A^T y) = (Ax)^T y = 0$ since $A^T y = 0$

On the other hand $(Ax)^T y \leq 0$ since $Ax < 0$ and $y \geq 0$

and the only way it is 0 is if $y = 0$ (since $Ax < 0$)

$\therefore$ The 2nd system is unsolvable. ☒

3(b) (1st system is unsolvable) $\Rightarrow$ (2nd system is solvable)

There is no $x \in \mathbb{R}^n$ satisfying $Ax < 0$

i.e $\nexists\, x \in \mathbb{R}^n$ satisfying $Ax \leq -e$

Consider the LP

$$\begin{aligned} &\text{Max } 0^T x \\ &\text{s.t } Ax \le -e \end{aligned}$$

This LP is infeasible

The dual LP is

$$\begin{aligned} &\text{MIN } -e^T y \\ &\text{s.t } A^T y = 0 \\ &\quad y \geq 0 \end{aligned}$$

By duality theory, the dual LP is either infeasible or unbounded.

However, $y = 0$ is a feasible solution in the dual LP.

∴ The dual LP is unbounded

∴ $\exists\ \bar{y} \in R^m$ s.t $-e^T \bar{y} < 0$, $A^T \bar{y} = 0$, $\bar{y} \geq 0$ Why?

i.e $e^T \bar{y} > 0$, $A^T \bar{y} = 0$, and $\bar{y} \geq 0$

Since $e^T \bar{y} > 0$ and $\bar{y} \geq 0$ we must have $\bar{y} \neq 0$

∴ $\exists\ \bar{y} \in R^m$ satisfying $A^T \bar{y} = 0$, $\bar{y} \geq 0$, and $\bar{y} \neq 0$

∴ The 2nd system is solvable ▨