MA/OR/IE 505-001: Linear Programming
Final Exam - May 1, 2007
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SOLUTIONS TO FINAL EXAM
PREPARED BY KARTIK

INSTRUCTIONS

1. Please write your name and student number clearly on the front page of the exam.
2. This has to be your own work. Cheating on the exam is not tolerated, and will fetch you a zero for the test.
3. TIME LIMIT: 3 hours
4. There are 6 pages and 5 questions on the exam. Each question appears on a different page. Read each question carefully.
5. The exam is worth 126 points of which 6 points are extra credit. The distribution of points is clearly indicated on the exam.
6. Solve each problem in sufficient detail in the space provided. Please use both sides of each page as needed.
7. Write clearly, including all the steps to the final solution. If I can't read it, you won't get credit.
8. Sources: Open book and class notes.
9. You may use an electronic calculator on the exam.
10. A useful formula for the inverse of 2×2 invertible matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

1. [26 **points**] Consider the following problem

$$\begin{array}{lrl} \max & Z = 4x_1 + \alpha x_2 + x_3 + \beta x_4 & \\ \text{s.t.} & 4x_1 + ax_2 + x_3 + bx_4 & \leq 5 \\ & 3x_1 + cx_2 + 2x_3 + dx_4 & \leq 4 \\ & x_1, x_2, x_3, x_4 & \geq 0. \end{array}$$

where the values of α, β, a, b, c, and d are unknown. Let x_5 and x_6 denote the slack variables for the two functional constraints. Now suppose Kartik has obtained the following dictionary at some iteration of the simplex method:

$$\begin{array}{rcl} x_2 & = & e + fx_1 + gx_3 - x_5 + x_6 \\ x_4 & = & h + ix_1 + jx_3 + x_5 - 2x_6 \\ \hline Z & = & \theta + kx_1 + lx_3 - x_5 - x_6 \end{array}$$

and you are unable to read some of the entries due to Kartik's bad handwriting :) Compute the following (in that order):

(a) [6 **points**] Find the values of a, b, c, and d.

(b) [4 **points**] Find the values of α and β.

(c) [2 **points**] Find the values e and h.

(d) [2 **points**] What is the objective value θ ?

(e) [10 **points**] Find all the remaining missing entries f, g, i, j, k, l in the dictionary.

(f) [2 **points**] Is the dictionary optimal? Why or why not?

1 (a) Since x_5 and x_6 are slack variable
We have $a_5 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $a_6 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Also $c_5 = c_6 = 0$

The general form of a dictionary is

$$\begin{array}{l} X_B = A_B^{-1} b - A_B^{-1} A_N X_N \\ \hline Z = C_B^T A_B^{-1} b + (C_N - A_N^T y)^T X_N \end{array} \qquad \text{where } A_B^T y = C_B$$

We have

$$-A_B^{-1} \begin{bmatrix} \uparrow & \uparrow \\ a_5 & a_6 \\ \downarrow & \downarrow \end{bmatrix} = -A_B^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \quad \therefore A_B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\therefore A_B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad \therefore \boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}}$$

1(b) We have

$$s_5 = c_5 - a_5^T y$$
$$s_6 = c_6 - a_6^T y$$
$$\therefore \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Also $\begin{bmatrix} s_5 \\ s_6 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$$\therefore \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore c_B = A_B^T y = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\therefore \boxed{\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}}$$

1(c) We have

$$x_B = \begin{bmatrix} e \\ h \end{bmatrix} = A_B^{-1} b = \begin{bmatrix} +1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\therefore \boxed{\begin{bmatrix} e \\ h \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}}$$

1(d) We have

$$\theta = c_B^T x_B = \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$= 3 + 6 = 9$$

1(e) We have

$$\begin{bmatrix} f & g \\ i & j \end{bmatrix} = -A_B^{-1} \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$
$$= -\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & -3 \end{bmatrix}$$

Also

$$\begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} s_1 \\ s_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} - \begin{bmatrix} a_1^T \\ a_3^T \end{bmatrix} y = \begin{bmatrix} 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} -3 \\ -2 \end{bmatrix}$$

1(f) DICTIONARY IS OPTIMAL

Since all the coeffs of nonbasic variables in the objective function are all nonpositive

2. [25 points] Consider the piecewise-linear minimization problem

$$\min_{x} \max_{i=1,\ldots,m} (a_i^T x + b_i) \tag{1}$$

where $x \in \mathbb{R}^n$, a_i is the ith row of $A \in \mathbb{R}^{m\times n}$, and b_i is the ith component of $b \in \mathbb{R}^m$.

(a) [10 points] Formulate the piecewise-linear minimization problem (1) as an LP. You must give the correct dimensions of all the variables to get full credit.

(b) [10 points] Form the dual of the LP that you obtained in part (a). You must again give the correct dimensions of all the dual variables to get full credit.

(c) [5 points] What are the complementary slackness conditions for this primal-dual pair of LPs. (You will NOT get any credit for writing the complementary slackness conditions on pages 62 and 63 of Chvátal for a general LP, so please confine your answer to the primal-dual LPs in parts (a) and (b))

2(a) We have

$$Z = \underset{x}{\text{MIN}} \ \underset{i=1,2,\ldots,m}{\text{Max}} \ (a_i^T x + b_i)$$

$$= \underset{x,t}{\text{MIN}} \ t \quad \text{s.t} \quad \underset{i=1,2,\ldots,m}{\text{Max}} (a_i^T x + b_i) \le t$$

$$= \underset{x,t}{\text{MIN}} \ t \quad \text{s.t} \quad (a_i^T x + b_i) \le t, \ i=1,2,\ldots,m$$

$$= \underset{x,t}{\text{MIN}} \ t \quad \text{s.t} \quad Ax - et \le -b$$

where $e \in \mathbb{R}^m$ is the all ones vector

↳ where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$

2(b) Dual is

$$W = \text{Max} \ -b^t y$$

$$\text{s.t} \quad \begin{bmatrix} A^T \\ -e^T \end{bmatrix} y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y \le 0$$

$\therefore$ Dual is

$$w = \text{Max } b^T y \quad \text{s.t} \quad A^T y = 0, \quad e^T y = 1, \quad y \geq 0$$

where $y \in R^m$

2(c) Complementary slackness conditions are

$$y_i (a_i^T x + b_i - t) = 0, \quad i = 1, 2, \ldots, m$$

3. [30 points] Consider the following problem

$$\begin{aligned} \max \quad & Z = c_1x_1 + c_2x_2 \\ \text{s.t.} \quad & 2x_1 - x_2 \le b_1 \\ & x_1 - x_2 \le b_2 \\ & x_1, x_2 \ge 0. \end{aligned} \tag{2}$$

Let x_3 and x_4 be the slack variables for the two functional constraints. When $c_1 = 3$, $c_2 = -2$, $b_1 = 30$, and $b_2 = 10$, the simplex method yields the following optimal dictionary:

$$\begin{array}{rcl} x_2 & = & 10 - x_3 + 2x_4 \\ x_1 & = & 20 - x_3 + x_4 \\ \hline Z & = & 40 - x_3 - x_4. \end{array}$$

(a) [5 points] What is the optimal solution for (2)? Is the optimal solution degenerate? Are there multiple optimal solutions? Construct B, A_B, and c_B from the optimal dictionary.

(b) [5 points] Construct the optimal dictionary for the dual problem directly from the optimal dictionary for the primal problem. What is the optimal solution for the dual problem?

(c) [10 points] Use sensitivity analysis to find the allowable range for c_1 and c_2 so that the previous primal optimal solution is still optimal in the new problem.

(d) [10 points] Use sensitivity analysis to find the allowable range for b_1 and b_2 so the previous primal optimal basis is still feasible in the new problem.

3(a) Opt solution is $X^* = \begin{pmatrix} 20 \\ 10 \\ 0 \\ 0 \end{pmatrix}$

Solution is nondegenerate since $X_B = \begin{pmatrix} 10 \\ 20 \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Unique optimal solution since all the coeffs of nonbasic variables in obj function row are negative

$B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $A_B = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$ $c_B = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

3(b) optimal dual dictionary is

$$\begin{array}{rcl} y_1 & = & 1 + y_4 + y_3 \\ y_2 & = & 1 - 2y_4 - y_3 \\ \hline -w & = & -40 - 10y_4 - 20y_3 \end{array}$$

Dual solution is $y^* = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

3(c) Let $c_B = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix}$

we have $A_B^T y = c_B$ $\therefore$ $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix}$

giving $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ -c_1 - 2c_2 \end{pmatrix}$

Now $s_N = c_N - A_N^T y$

$= - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 + c_2 \\ -c_1 - 2c_2 \end{bmatrix} = \begin{bmatrix} -c_1 - c_2 \\ c_1 + 2c_2 \end{bmatrix}$

Previous solution is still optimal as long as $s_N \leq 0$

i.e $-c_1 - c_2 \leq 0$ and $c_1 + 2c_2 \leq 0$

i.e as long as $\boxed{-c_2 \leq c_1 \leq -2c_2}$

3(d) Let $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

we have

$x_B = A_B^{-1} b = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$= \begin{bmatrix} b_1 - 2b_2 \\ b_1 - b_2 \end{bmatrix}$

Previous optimal basis is still feasible as long as $x_B \geq 0$

i.e $b_1 - 2b_2 \geq 0$ and $b_1 - b_2 \geq 0$

i.e $\boxed{b_1 \geq \text{Max}\{b_2, 2b_2\}}$

4. [25 points] Let $A \in \mathbb{R}^{m \times n}$ be a given matrix. Use LP duality to show the following:

(a) [15 points] If every $x \in \mathbb{R}^n$ satisfying $Ax \geq 0$ and $x \geq 0$ must have $x_1 = 0$; then there exists a vector $y \in \mathbb{R}^m$ satisfying $A^T y \leq 0$, $y \geq 0$, and $a_1^T y < 0$ where a_1 is the first column of the matrix A. (Hint: Consider the LP where one maximizes x_1 subject to $Ax \geq 0$ and $x \geq 0$).

(b) [10 points] Show the converse of (a): Suppose there exists a vector $y \in \mathbb{R}^m$ satisfying $A^T y \leq 0$, $y \geq 0$, and $a_1^T y < 0$; then every vector $x \in \mathbb{R}^n$ satisfying $Ax \geq 0$ and $x \geq 0$ must have $x_1 = 0$.

4(a) Consider the LP

$$z = \text{Max } x_1 \quad \text{s.t } Ax \geq 0, \; x \geq 0 \qquad = \qquad \text{Max } \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{s.t } Ax \geq 0, \; x \geq 0$$

Since every $x \in \mathbb{R}^n$ satisfying $Ax \geq 0$ and $x \geq 0$ also has $x_1 = 0$, the above LP is optimal with an optimal objective value of 0.

$\therefore$ By duality theory, the dual LP

$$W = \text{MIN } 0^T y \quad \text{s.t } A^T y \geq \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; y \leq 0$$

is also optimal with an objective value of 0

$\therefore \exists\, y$ satisfying $a_1^T y \geq 1$, $a_2^T y \geq 0$, $\ldots$, $a_m^T y \geq 0$ and $y \leq 0$

$\therefore \exists\, y$ satisfying $a_1^T y \leq -1$, $a_2^T y \leq 0$, $\ldots$, $a_m^T y \leq 0$ and $y \geq 0$ (Replacing y with $-y$)

$\therefore \exists\, y$ satisfying $A^T y \leq 0$, $y \geq 0$, and $a_1^T y < 0$

4(b) We are given a $y \in \mathbb{R}^m$ satisfying $A^T y \leq 0$, $y \geq 0$, and $a_1^T y < 0$

ie we have a $y \in \mathbb{R}^m$ satisfying $A^T y \leq 0$, $y \geq 0$ and $a_1^T y \leq -1$ (why?)

$\therefore$ We have a $y \in \mathbb{R}^m$
Satisfying $A^T y \geq 0$, $y \leq 0$, and $a_1^T y \geq 1$

$\therefore$ The LP

$$w = \text{MIN } 0^T y$$
$$\text{s.t } A^T y \geq \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
$$y \leq 0$$

is feasible and optimal with a objective value of 0.

By duality theory, the dual LP

$$z = \text{Max } x_1$$
$$\text{s.t } Ax \geq 0$$
$$x \geq 0$$

is optimal with a objective value of 0 too

i.e every $x \in \mathbb{R}^n$ satisfying $Ax \geq 0$, $x \geq 0$ must also satisfy $x_1 = 0$.

5. [20 points] Consider the linear program

$$\begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & \leq & b \\ & Ex & \leq & d \\ & x & \geq & 0 \end{array} \qquad (3)$$

where $c, x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $E \in \mathbb{R}^{q \times n}$, $b \in \mathbb{R}^m$, and $d \in \mathbb{R}^q$. Let $y \in \mathbb{R}^m$ and $w \in \mathbb{R}^q$ denote the dual variables corresponding to the constraints $Ax \leq b$ and $Ex \leq d$ in (3) respectively. Let x^* be an optimal solution to (3) and let (y^*, w^*) be an optimal solution to its dual.

(a) [5 points] Write down the optimality conditions (primal feasibility, dual feasibility, and complementary slackness) that are satisfied by x^* and (y^*, w^*).

(b) [5 points] Consider the LP

$$\begin{array}{lrcl} \max & (c - A^T y^*)^T x & & \\ \text{s.t.} & Ex & \leq & d \\ & x & \geq & 0. \end{array} \qquad (4)$$

Show that x^* and w^* are also optimal solutions for (4) and its dual (Hint: Show that x^* and w^* satisfy the optimality conditions for (4)).

(c) [5 points] Illustrate the property from part (b) with the optimal solution $x^* = (2, 1)$ and optimal dual solution $y^* = \frac{1}{2}$ and $w^* = (\frac{1}{2}, 0)$ to the LP

$$\begin{array}{lrcl} \max & x_1 & & \\ \text{s.t.} & x_1 + x_2 & \leq & 3 \\ & x_1 - x_2 & \leq & 1 \\ & x_2 & \leq & 2 \\ & x_1, x_2 & \geq & 0. \end{array} \qquad (5)$$

where $x_1 + x_2 \leq 3$ is in $Ax \leq b$, and $x_1 - x_2 \leq 1$, $x_2 \leq 2$ are in $Ex \leq d$ respectively. You will solve (3) and (4) for this example graphically.

(d) [5 points] Show that an optimal solution to (4) need not be an optimal solution to (3) by looking at the LP (5). Under what conditions is an optimal solution to (4) also an optimal solution to (3)?

5(a) Dual to (3) is

$w = \text{MIN } b^T y + d^T w$

$\text{s.t } A^T y + E^T w \geq c$

$y \geq 0, w \geq 0$

5(a) (CONTINUED) Optimality conditions satisfied by x^* and (y^*, w^*) include

$Ax^* \le b$, $Ex^* \le d$, $x^* \geq 0$ (PRIMAL FEAS)

$A^Ty^* + E^Tw^* \geq c$, $y^* \geq 0$, $w^* \geq 0$ (DUAL FEAS)

$y_i^*(a_i^Tx^* - b_i) = 0$, $i = 1, 2, \ldots, m$ (COMPLEMENTARY SLACKNESS)

$w_j^*(e_j^Tx^* - d_j) = 0$, $j = 1, 2, \ldots, q$

$x_k^*(A^Ty^* + E^Tw^* - c^*)_k = 0$, $k = 1, 2, \ldots, n$ $\hookrightarrow$ ①

5(b) The dual to (4) is

Min d^Tw

s.t $E^Tw \geq (c - A^Ty^*)$

$w \geq 0$

Opt conditions satisfied by an optimal x^* and w^* to (4) include $Ex^* \le d$, $E^Tw^* \geq (c - A^Ty^*)$, $w_j^*(e_j^Tx^* - d_j) = 0$, $j = 1, 2, \ldots, q$

$x^* \geq 0$, $w^* \geq 0$, $x_k^*(A^Ty^* + E^Tw^* - c^*)_k = 0$, $k = 1, 2, \ldots, n$

which are already satisfied by the optimal x^* and w^* to ③ (from ①)

$\therefore$ x^* and w^* are also optimal solutions to (4) and its dual respectively

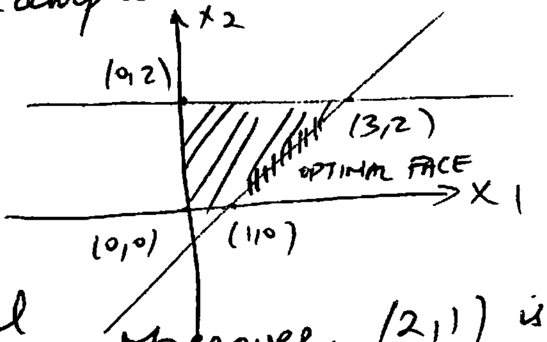
5(c) The LP (4) for example 5 is

Max $\frac{1}{2}x_1 - \frac{1}{2}x_2$

s.t $x_1 - x_2 \le 1$

$x_2 \le 2$

$x_1, x_2 \geq 0$



This LP has multiple optimal solutions (line segment joining (1,0) to (3,2))

Moreover, (2,1) is one of these optimal solutions

5(d) Note that (1,0) and (3,2) are not optimal solutions to ③ (these are extreme pt solutions to (4))

An optimal solution to (4) will also be an optimal soln to ③ if and only if it is UNIQUE

if and only if it satisfies $y_j^*(a_j^Tx^* - b_j) = 0$, $j = 1, 2, \ldots, m$