MA/IE/OR 505 SPRING 2007

SOLUTIONS TO REVIEW PROBLEMS FOR FINAL EXAM

(PREPARED BY KARTIK)

PROBLEM (1)

[redacted]] Consider the linear programming problem

$$\begin{array}{lrcl} \min & c^T x - b^T y & & \\ \text{s.t.} & Ax & \geq & b \\ & -A^T y & \geq & -c \\ & x & \geq & 0 \\ & y & \geq & 0, \end{array}$$

where A is a $m \times n$ matrix, $b \in I\!R^m$ and $c \in I\!R^n$.

(a) [20 points] Write down the dual and show that this LP is *self-dual*. Use this feature to show that the original LP is either infeasible or optimal.

(b) [10 points] What are the optimality conditions (primal feasibility, dual feasibility, and complementary conditions) for this LP?

(c) [Extra Credit: 10 points] If the LP is optimal, then show that the optimal objective value is zero.

1 (a) Let $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ be the dual variables corresponding to the first two sets of constraints

DUAL IS

$$\begin{array}{ll} \text{Max} & b^T u - c^T v \\ \text{s.t} & \begin{bmatrix} A^T & 0 \\ 0 & -A \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \leq \begin{bmatrix} c \\ -b \end{bmatrix} \\ & u \geq 0, \quad v \geq 0 \end{array}$$

$$= \begin{array}{ll} -\text{MIN} & c^T v - b^T u \\ \text{s.t} & A^T u \leq c \\ & -Av \leq -b \\ & u \geq 0, v \geq 0 \end{array} = \begin{array}{ll} -\text{MIN} & c^T u - b^T v \\ \text{s.t} & -A^T u \geq c \\ & Av \geq b \\ & u \geq 0, v \geq 0 \end{array}$$

$\therefore$ The LP is self dual with $x \Leftrightarrow v$, $y \Leftrightarrow u$

By the duality theory of LP, a self dual LP can only be infeasible or OPTIMAL (because primal and dual are the same problem)

$\therefore$ The original LP is either infeasible or OPTIMAL.

1(b) The optimality conditions are

$$Ax \geq b \quad -A^T y \geq -c \quad x \geq 0,\ y \geq 0 \qquad \left(\text{PRIMAL FEAS}\right)$$

$$-A^T u \geq -c \quad Av \geq b \quad u \geq 0,\ v \geq 0 \qquad \left(\text{DUAL FEAS}\right)$$

$u_i\,(Ax-b)_i = 0,\quad i = 1,2,\dots,m$

$v_j\,(c-A^Ty)_j = 0,\quad j = 1,2,\dots,n$

$x_k\,(c-A^Tu)_k = 0,\quad k = 1,2,\dots,n$

$y_l\,(Av-b)_l = 0,\quad l = 1,2,\dots,m$

Using the self dual feature of the original LP i.e $x \Leftrightarrow v$, $y \Leftrightarrow u$

we can eliminate half of these conditions and rewrite the optimality conditions as follows:—

$$\boxed{\begin{array}{l} Ax \geq b \\ -A^Ty \geq -c \\ x \geq 0,\ y \geq 0 \\ y_i\,(Ax-b)_i = 0,\quad i = 1,2,\dots,m \\ x_j\,(c-A^Ty)_j = 0,\quad j = 1,2,\dots,n \end{array}}$$

1(c) The last two conditions give $y^T(Ax-b) = 0$ and $x^T(c-A^Ty) = 0$ (BY SUMMING these two sets of constraints respectively)

This gives
$b^Ty = y^T(Ax) = (Ax)^Ty = c^Tx$

$\therefore$ If x^* and y^* are optimal solutions to the original LP
$c^Tx^* - b^Ty^* = 0$, i.e, this LP has an optimal objective value of zero.

PROBLEM (2) SOLUTION TO A SIMILAR PROBLEM

~~4. [30 points]~~ Consider the following optimal dictionary

$$\begin{array}{rcl} x_5 & = & 10 - 3x_2 - x_3 - x_4 \\ x_1 & = & 6 - x_2 - x_3 - x_4 \\ \hline Z & = & 12 - 3x_2 - x_3 - 2x_4 \end{array}$$

of an LP (maximization problem, constraints are of the $\leq$ type). It is known that x_4 and x_5 are the slack variables in the first and second constraints of the original problem.

(a) [20 points] Write down the original LP and its dual.

(b) [5 points] Obtain the optimal solutions to the primal and dual LPs directly from the dictionary.

(c) [5 points] Are there alternate primal or dual optimal solutions? Give reasons for your answer.

2 (a) We have

$$x_B = A_B^{-1}b - A_B^{-1}A_N x_N$$

$$z = c_B^T A_B^{-1} b + (c_N - A_N^T y)^T x_N$$

We have $x_B = \begin{bmatrix} x_5 \\ x_1 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}$ $A_B^{-1}A_N = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$x_N = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $z = c_B^T A_B^{-1} b = 12$

and $s_N = (c_N - A_N^T y) = \begin{bmatrix} -3 \\ -1 \\ -2 \end{bmatrix}$

Note that

$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \end{bmatrix}$ with $a_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $a_5 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Also $c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 0 \\ 0 \end{bmatrix}$

(since x_4, x_5 are SLACK VARIABLES in the problem)

Now $s_N = (c_N - A_N^T y) = \begin{bmatrix} -3 \\ -1 \\ -2 \end{bmatrix}$

gives $y_1 = 2$ (since $c_4 = 0$ and $a_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$)

$\therefore$ $\boxed{y = \begin{bmatrix} 2 \\ 0 \end{bmatrix}}$

Also $y_2 = 0$ from $A_B^T y = c_B$ (since $c_5 = 0$ and $a_5 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$)

Consider

$A_B^{-1} A_N = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$\therefore$ $\begin{bmatrix} a_{12} & a_{13} & 1 \\ a_{22} & a_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_{11} \\ 1 & a_{21} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} a_{11} & a_{11} & a_{11} \\ (a_{21}+3) & (a_{21}+1) & (a_{21}+1) \end{bmatrix}$

$A_N = A_B \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} a_2 & a_3 & a_4 \end{bmatrix} = \begin{bmatrix} a_{12} & a_{13} & 1 \\ a_{22} & a_{23} & 0 \end{bmatrix}$

$= \begin{bmatrix} a_5 & a_1 \end{bmatrix} = \begin{bmatrix} 1 & a_{11} \\ 0 & a_{21} \end{bmatrix}$

$\therefore$ $\boxed{a_{11} = 1}$ $\boxed{a_{21} = -1}$

$\boxed{a_{12} = a_{11} = 1}$ $\boxed{a_{13} = a_{11} = 1}$

$\boxed{a_{23} = -1+1 = 0}$ $\boxed{a_{22} = (a_{21}+3) \ldots}$

$$\therefore A = \begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ -1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

Now $b = A_B x_B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_5 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

we have

$s_N = (c_N - A_N^T y) = \begin{bmatrix} -3 \\ -1 \\ -2 \end{bmatrix}$

$\therefore \begin{bmatrix} c_2 \\ c_3 \\ c_4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ -2 \end{bmatrix}$

$\therefore \boxed{c_2 = -3 + 2 = -1}$

$\boxed{c_3 = -1 + 2 = 1}$

Also $c_1 = y_1 - y_2$ (from the first ... of $A_B^T y$ ...) $= 2 - 0 = 2$

$\therefore \boxed{c = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}}$

$\therefore$ The LP is

Max $2x_1 - x_2 + x_3$

s.t. $x_1 + x_2 + x_3 \le 6$

$-x_1 + 2x_2 \le 4$

$x_1, x_2, x_3 \geq 0$

Dual is

Max $6y_1 + 4y_2$

st $y_1 - y_2 \geq 2$

$y_1 + 2y_2 \geq -1$

$y_1 \geq 1$

$y_1, y_2 \geq 0$

2(b) $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}$

$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}$

2(c) Both solutions are nondegenerate which implies that the corresponding dual solutions are UNIQUE.

PROBLEM (3) :— WORK IT OUT YOURSELF

PROBLEM (4)

OPTIMAL DICTIONARY FOR FINAL problem is

(a)

$$
\begin{aligned}
y_3 &= 3 + y_4 + 2y_6 \\
y_5 &= 2 - y_4 + 3y_6 \\
y_1 &= 1 + y_4 - y_6 \\
y_2 &= 1 - y_4 + 2y_6 \\
\hline
-w &= -9 - y_4 - 3y_6
\end{aligned}
$$

OPT SOLUTION IS $y^* = \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

(b) (i) We have

$$A_B = \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 2 \end{bmatrix} \qquad C_B = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 5 \end{bmatrix} \text{ etc}$$

COMPUTE

$$w = (A_B^T)^{-1} C_B = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 7 \end{bmatrix}$$

Also,

$$N = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$C_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ etc}$$

$$A_N = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Now compute $S_N = (C_N - A_N^T w)$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 0 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -7 \end{bmatrix}$$

Since the reduced cost for y_4 is 2 (POSITIVE)

our earlier solution $y^* = \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is no longer optimal.

b (ii) Compute

$$y_B = A_B^{-1} b = \begin{bmatrix} -8 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

Since $y_B \ngeq 0$, the earlier solution $y^* = (1,1)$ is no longer feasible in new problem.

b (iii) Since $2y_1^* + 4y_2^* = 2(1) + 4(1)$
$= 6 \ngeq 7$

the earlier solution $y^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is no longer feasible in the new problem.

PROBLEM (5) WORKOUT YOURSELF

PROBLEM (6)

~~5. [30 points]~~ Consider the LP

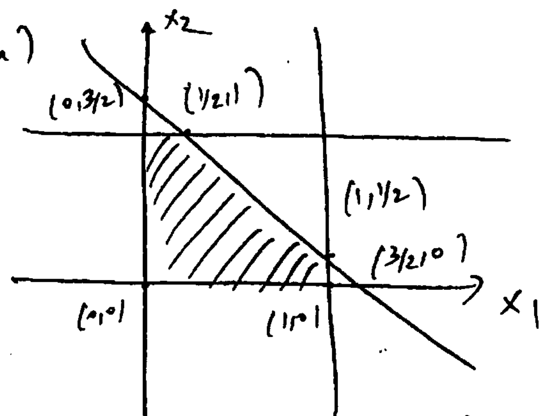
$$\begin{array}{lrcl} \max & x_1 + x_2 & & \\ \text{s.t.} & x_1 + x_2 & \le & \frac{3}{2} \\ & x_1, x_2 & \le & 1 \\ & x_1, x_2 & \ge & 0. \end{array}$$

(a) [5 points] Sketch the feasible region of the LP and solve it geometrically. What is the optimal solution to this LP?

(b) [20 points] Solve the LP using the decomposition approach by treating the first constraint as the coupling constraint and the remaining constraints as the constraints in the subproblem. Start the decomposition approach in the usual way with the extreme point $(0, 0)$. You will need 2 iterations to solve the problem.

(c) [5 points] What is the optimal solution generated by the decomposition scheme for the original LP? Is this an extreme point solution?

6(a)



The optimal solution set is the line segment joining $(1/2,1)$ and $(1,1/2)$ including these end points

6(b) write the problem as

$$\begin{array}{ll} \text{Max} & c^T x + d^T w \\ \text{s.t} & Ax + Dw = b \\ & B_1 x = d_1 \\ & x \ge 0,\ w \ge 0 \end{array}$$

In our case

$A = [1\ 1\ 0\ 0]$, $D = [1]$, $b = 3/2$, $d_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $c = \begin{bmatrix}1\\1\\ \vdots \end{bmatrix}$, $d = 0$

$B_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ $w = [x_5]$ etc

Our master problem is

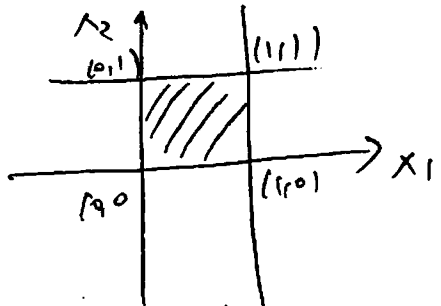
$$\text{Max} \sum_{j \in J_1} (c^T x^j)\lambda^j + d^T w$$

$$\text{s.t} \sum_{j \in J_1} \begin{bmatrix} Ax^j \\ 1 \end{bmatrix} \lambda^j + \begin{bmatrix} D \\ 0 \end{bmatrix} w = \begin{bmatrix} b \\ 1 \end{bmatrix} \to \bar{b} \text{ (say)}$$

$$\lambda^j \ge 0,\ j \in J_1$$

$$w \ge 0$$

$\to$ we solve this problem using the revised simplex method with column generation



FEASIBLE region of subproblem $= \left\{ x \,\middle|\, \begin{array}{l} B_1 x = d_1 \\ x \ge 0 \end{array} \right\}$

Start with

$$A_B = \begin{bmatrix} A\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix} & 1 \\ 1 & 0 \end{bmatrix}, \quad c_B = \left[c^T\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}, 0 \right]^T$$

we have

$$A_B x_B = \bar{b} = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \quad \therefore x_B = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix}$$

Step 2:- Solve $A_B^T \begin{bmatrix} y \\ z_1 \end{bmatrix} = c_B$ for y^* and z_1^*

This gives $y^* = 0$ and $z_1^* = 0$

Solve

$$Q_1^* = \max (c - A^T y^*)^T x \quad \text{s.t. } B_1 x = d_1, \; x \ge 0$$

$$= \max x_1 + x_2 \quad \text{s.t. } x_1 + x_3 = 1, \; x_2 + x_4 = 1, \; x_1, \dots, x_4 \ge 0$$

Opt soln is $x^* = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ for a obj value of $Q_1^* = 2$

Since $s_1^* = Q_1^* - z_1^* = 2 > 0$

we have $a_N = \begin{bmatrix} A\begin{bmatrix}1\\1\\0\\0\end{bmatrix} \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is the new ent column $\Leftarrow$

Step 3:- Solve $A_B d = a_N \Rightarrow d = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\theta = \min \left\{ \frac{x_{B1}}{d_1}, \frac{x_{B2}}{d_2} \right\} = \min \{1, 3/4\} = 3/4$$

Step 4:- $x_B(1) = x_B(1) - \theta d_1 = 1 - 3/4 = 1/4$

$x_B(2) = \theta = 3/4$

$$\therefore x_B = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix} \quad A_B = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad c_B = \left[0 \;\; c^T\begin{bmatrix}1\\1\\0\\0\end{bmatrix} \right]^T = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

ITER 2:-

Solve $A_B^T \begin{bmatrix} y \\ z_1 \end{bmatrix} = c_B$

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$z_1 = 0$ and $y = 1$

This gives the sub $Q_1^* = \max 0^T x$ s.t. $x_1 + x_3 = 1$, $x_2 + x_4 = 1$, ... $\therefore Q_1^* = 0$

$\therefore s_1^* = Q_1^* - z_1^* = 0$ and we are optimal.

6(c) The opt soln in the space of orig var $= 1/4 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 3/4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 3/4 \end{pmatrix}$ NOT AN EXTREME PT

PROBLEM (7)

~~1. [20 points]~~ Kartik is planning to spend the day at the races. He will have b\$ available for betting. On the day of his proposed visit, there is only one race planned in which horses numbered 1, 2, ..., k are competing. Kartik's gross payoff is α_i\$ (for every dollar he bets on the ith horse) if this horse comes first in the race, and 0 \$ otherwise. The numbers $\alpha_1, \ldots, \alpha_k$ are known positive integers. Kartik is, of course, allowed to bet any nonnegative amount on any number of horses. His problem is to determine how much to bet on each of the horses in the race so as to maximize his minimum gross payoff, irrespective of whichever horse comes first in the race.

(a) [20 points] Formulate this problem as an LP.

(b) [10 points] Write down the special case of this problem when $b = 100$, $k = 5$, and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (2, 5, 1.5, 7, 3)$.

7(a) Let $x_1, \ldots x_k$ be the amount in \$ that Kartik bets on the k horses

We have $x_1, x_2, \ldots, x_k \geq 0$

Also $x_1 + x_2 + \cdots + x_k \leq b$ (BUDGET CONSTRAINT)

The payoffs are $\alpha_1 x_1$ if horse 1 wins the race

$\vdots$

$\alpha_k x_k$ if horse k wins the race

Kartik's minimum payoff $= P = \min_{i=1,2,\ldots,k} \{\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_k x_k\}$

Since Kartik wants to maximize his minimum payoff his problem is

$$\max_{x_1, \ldots, x_k} \quad P = \min_{i=1,2,\ldots,k} \{\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_k x_k\}$$

$$\text{s.t.} \quad x_1 + x_2 + \cdots + x_k \leq b$$

$$x_1, \ldots, x_k \geq 0$$

i.e $\displaystyle \max_{x_1,\dots,x_k} \ \min_{i=1,2,\dots,k} \{\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_k x_k\}$

s.t $x_1 + x_2 + \cdots + x_k \le b$

$x_1, \dots, x_k \geq 0$

Introducing a new variable t, this problem can be reformulated as the following LP

$$\boxed{\begin{array}{lll} \displaystyle\max_{x_1,\dots,x_k} & t & \\ & \text{s.t} & \alpha_1 x_1 \geq t \\ & & \vdots \\ & & \alpha_k x_k \geq t \\ & & x_1 + \cdots + x_k \le b \\ & & x_1, \dots, x_k \geq 0 \end{array}}$$

7(b) For our special case, the LP is

$$\begin{array}{lll} \displaystyle\max_{x, t} & t & \\ & \text{s.t} & 2x_1 \geq t \\ & & 5x_2 \geq t \\ & & 1.5x_3 \geq t \\ & & 7x_4 \geq t \\ & & 3x_5 \geq t \\ & & x_1 + x_2 + \cdots + x_5 \le 100 \\ & & x_1, \dots, x_5 \geq 0 \end{array}$$

i.e $(2x_1 - t) \geq 0$ etc

PROBLEM (8)

~~5. [25 points]~~ Solve the following integer programming problem

$$
\begin{array}{llcl}
\max & Z = -x_1 + 2x_2 & & \\
\text{s.t.} & -4x_1 + 6x_2 & \leq & 9 \\
& x_1 + x_2 & \leq & 4 \\
& x_1, x_2 & \geq & 0 \\
& x_1, x_2 & \text{integer.} &
\end{array}
$$

using branch and bound. You must follow these details carefully:

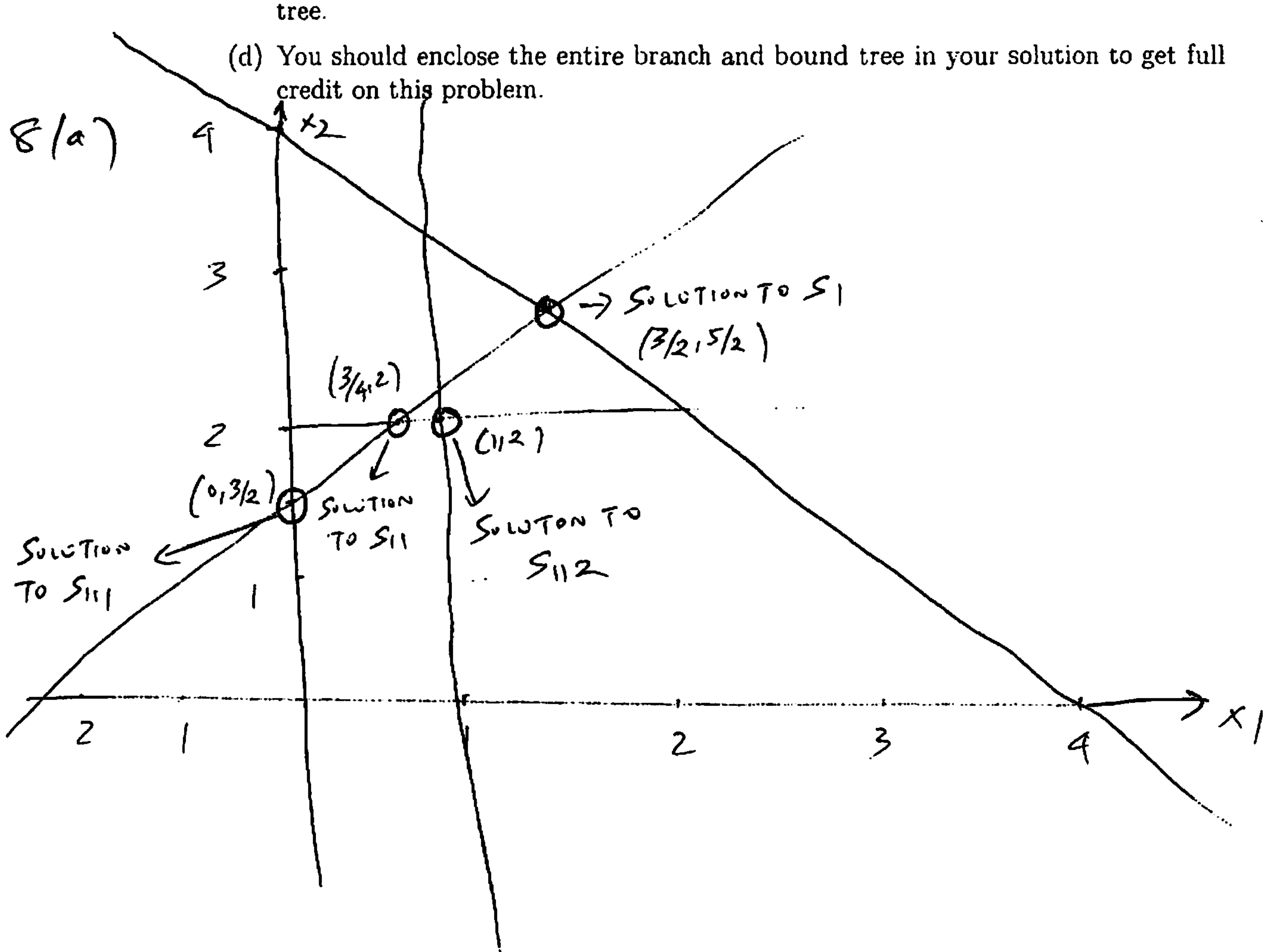
(a) Solve the linear programming relaxations at each node of the branch and bound tree graphically.

(b) In the 1st iteration, you will have a choice to branch on either x_1 or x_2. Branch on x_2.

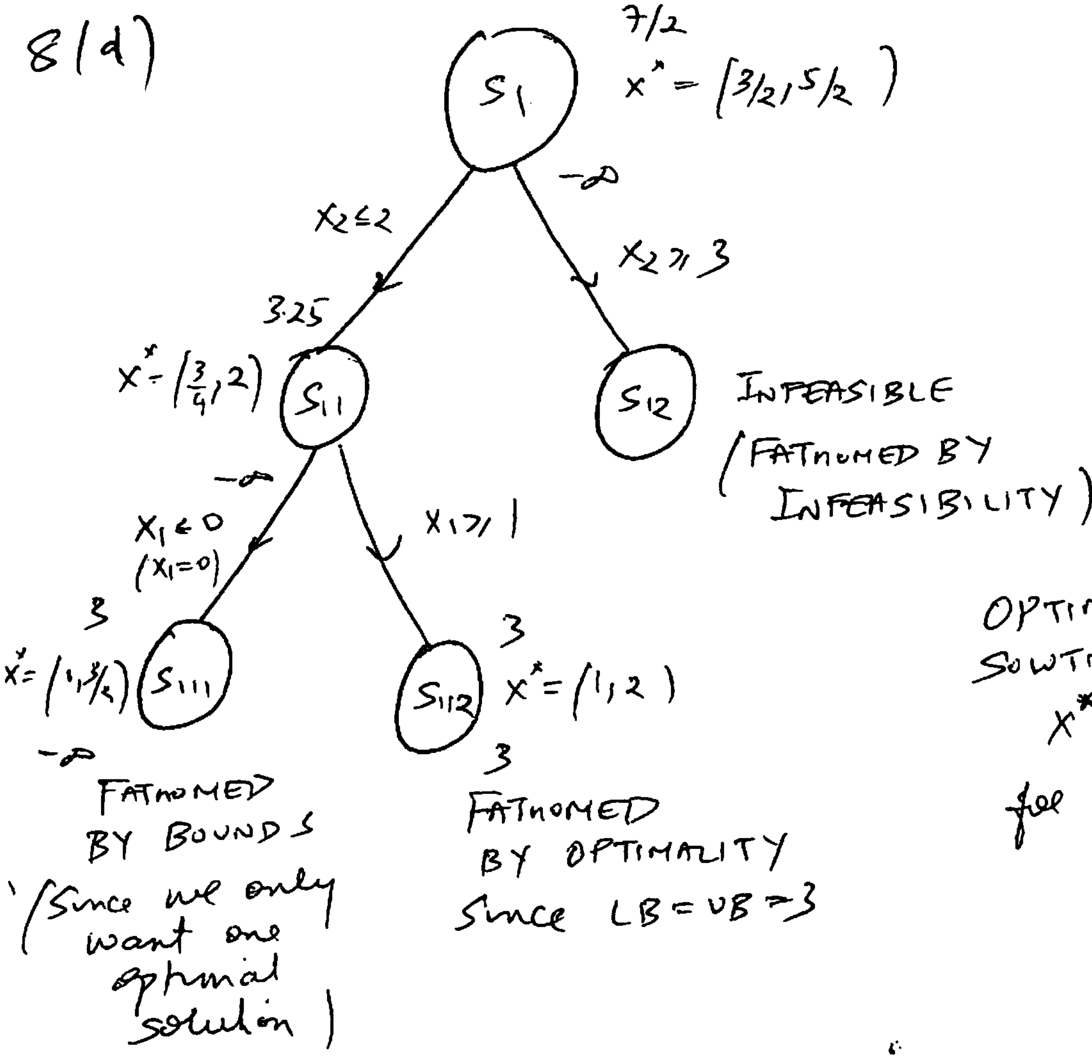
(c) Examine the nodes in the branch and bound tree in a *breadth-first* fashion, i.e., solve all the subproblems at one level before proceeding to the next level of the tree.

(d) You should enclose the entire branch and bound tree in your solution to get full credit on this problem.

8(a)



8(d)



OPTIMAL SOLUTION IS

$x^* = [1, 2]$

for a objective value of

$-1 + 4 = \boxed{3}$