

MA/IE/OR 505-001: Linear Programming
Spring 2007
Final Exam Review
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INSTRUCTIONS

The questions on the final will be similar to these review problems. Work out each problem in sufficient detail. If you have any questions, please send me an email. I will post the solutions to selected problems on the course webpage on Friday.

1. Consider the linear programming problem

$$\begin{array}{ll} \min & c^T x - b^T y \\ \text{s.t.} & Ax \geq b \\ & -A^T y \geq -c \\ & x \geq 0 \\ & y \geq 0, \end{array}$$

where A is a $m \times n$ matrix, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.

- (a) What is the dual to this LP? Use the dual to show that the original LP is either infeasible or optimal.
 - (b) Write down the complementary slackness conditions.
 - (c) If the LP is optimal, then show that the optimal objective value is zero.
2. Consider the following optimal dictionary

$$x_3 = \frac{5}{2} - \frac{1}{2}x_2 - \frac{1}{4}x_4 + 0x_5$$

$$x_1 = 2 + \frac{1}{2}x_2 + \frac{1}{6}x_4 + \frac{1}{3}x_5$$

$$Z = 35 - 2x_2 - 4x_4 - 2x_5,$$

of an LP (maximization problem, constraints are of the \leq type). It is known that x_4 and x_5 are the slack variables in the first and second constraints of the original problem.

- (a) Write down the original LP and its dual.

- (b) Obtain the optimal solutions to the primal and dual LPs directly from the dictionary.
- (c) Are there alternate primal and dual optimal solutions? Give reasons for your answer.

3. Consider the linear program

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. Suppose, we find a vector $d \in \mathbb{R}^n$ satisfying $Ad = 0$, $d \geq 0$, and $c^T d < 0$. Then for any feasible vector x in (1), show that $x + \alpha d$ is also feasible for all $\alpha > 0$. Also, show that $c^T(x + \alpha d) < c^T x$. Deduce that if such a vector d exists, then the LP (1) is unbounded.

4. Consider the following problem

$$\begin{aligned} \min \quad & W = 5y_1 + 4y_2 \\ \text{s.t.} \quad & 4y_1 + 3y_2 \geq 4 \\ & 2y_1 + y_2 \geq 3 \\ & y_1 + 2y_2 \geq 1 \\ & y_1 + y_2 \geq 2 \\ & y_1, y_2 \geq 0. \end{aligned} \tag{2}$$

Because this primal problem has more functional constraints than variables, suppose that the simplex method has been applied directly to its dual problem. If we let x_5 and x_6 denote the slack variables for the dual problem, the final optimal dictionary for the dual problem is

$$\begin{aligned} x_2 &= 1 - x_1 + x_3 - x_5 + x_6 \\ x_4 &= 3 - 2x_1 - 3x_3 + x_5 - 2x_6 \\ \hline Z &= 9 - 3x_1 - 2x_3 - x_5 - x_6. \end{aligned} \tag{3}$$

- (a) Construct the optimal dictionary for the primal problem (2) directly from the optimal dictionary (3) for the dual problem.
- (b) For each of the following independent changes in the primal problem, use sensitivity analysis to determine whether the current solution to the primal problem (2) is still feasible and whether it is still optimal. Do not reoptimize the new problem. Then check your answer by a direct graphical analysis of the primal problem.
- i. Change the objective function to $W = 3y_1 + 5y_2$.
 - ii. Change the right hand sides of the functional constraints to 3, 5, 2, and 3, respectively.
 - iii. Change the first constraint to $2y_1 + 4y_2 \geq 7$.

5. Solve the following problem

$$\begin{array}{ll} \max & -4x_1 - 6x_2 - 5x_3 \\ \text{s.t.} & 2x_1 + 3x_3 \geq 3, \\ & 3x_2 + 2x_3 \geq 6, \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

by the dual simplex method.

6. Consider the LP

$$\begin{array}{ll} \max & x_1 + x_2 \\ \text{s.t.} & x_1 + x_2 \leq \frac{3}{2} \\ & x_1, x_2 \leq 1 \\ & x_1, x_2 \geq 0. \end{array}$$

- Sketch the feasible region of the LP and solve it geometrically. What is the optimal solution to this LP?
 - Solve the LP using the decomposition approach treating the first constraint as the coupling constraint and the remaining constraints as the constraints in the subproblem. Restart the decomposition approach in the usual way with the extreme point $(0, 0)$. You will need 2 iterations to solve the problem.
 - What is the optimal solution generated by the decomposition scheme for the original LP? Is this an extreme point solution?
7. Kartik is planning to spend the day at the races. He will have b \$ available for betting. On the day of his proposed visit, there is only one race planned in which horses numbered $1, 2, \dots, k$ are competing. If Kartik bets a dollar on the i th horse, his gross payoff is α_i \$ (for every dollar bet) if this horse comes first in the race, and 0 \$ otherwise. The numbers $\alpha_1, \dots, \alpha_k$ are known positive integers. Kartik is, of course, allowed to bet any nonnegative amount on any number of horses. His problem is to determine how much to bet on each of the horses in the race so as to maximize his minimum gross payoff, irrespective of whichever horse comes first in the race. Formulate this problem as an LP. Write down the special case of this problem when $b = 100$, $k = 5$, and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (2, 5, 1.5, 7, 3)$.

8. Solve the following integer programming problem

$$\begin{array}{ll} \max & Z = -x_1 + 2x_2 \\ \text{s.t.} & -4x_1 + 6x_2 \leq 9 \\ & x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \text{ integer.} \end{array}$$

using branch and bound. You must follow these details carefully:

- Solve the linear programming relaxations at each node of the branch and bound tree graphically.

- (b) In the 1st iteration, you will have a choice to branch on either x_1 or x_2 . Branch on x_2 .
- (c) Examine the nodes in the branch and bound tree in a *breadth-first* fashion, i.e., solve all the subproblems at one level before proceeding to the next level of the tree.
- (d) You should enclose the entire branch and bound tree in your solution to get full credit on this problem.