MA/OR/IE 505-001: Linear Programming
Class Project
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1 Introduction

Consider the LP

$$\begin{array}{lrcl} \min & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array} \tag{1}$$

with $A \in I\!R^{m \times n}$, $b \in I\!R^m$, and $x, c \in I\!R^n$. We will assume that the number of columns n of matrix A is so large that it is impossible to generate and store the matrix A in memory. The revised simplex method in Chapter 7 of Chvátal can be appropriately modified to solve linear programs of the form (1); the resulting method is known as the *column generation* method where columns of A are generated as needed. One such LP arises in the cutting stock problem (see Chapter 13 in Chvátal). A short description of the cutting stock problem is also given below. Our goal in this project is to develop a computer implementation of the column generation method in MATLAB to solve the cutting stock problem.

You will work in groups of 3 students. The entire project is due in class on Tuesday, the 5th of December. If you have questions on any parts of the project, you are welcome to send me email or come and see me.

2 Cutting Stock Problem and Column Generation Algorithm

Consider a paper company that has a supply of large rolls of paper of width r inches. However, customer demand is for smaller widths of paper; in particular b_i rolls of width w_i inches, $i = 1, \ldots, m$, need to be produced. We assume that $w_i \leq r$, $i = 1, \ldots, m$. Smaller rolls are obtained by slicing a large roll in a certain way, called a *pattern*. For example, consider a large roll of width $r = 100$; and the following $m = 4$ consumer demand widths: $w_1 = 45$, $w_2 = 36$, $w_3 = 31$, $w_4 = 14$ respectively. The large roll can be cut into one roll of width $w_1 = 45$, one roll of width $w_3 = 31$, and one roll of width $w_4 = 14$ with a waste of 10 inches. We represent this pattern by the vector $(1, 0, 1, 1)$. In general, a pattern, say the jth pattern, can be represented by a column vector a_j whose ith entry a_{ij} indicates the number of rolls of width w_i inches that are produced by that pattern. For a vector

$a_j = (a_{1j}, a_{2j}, \dots, a_{mj})^T$ to represent a feasible pattern, we must have

$$a_j \geq 0, \ \sum_{i=1}^{m} a_{ij} w_i \leq r, \ \text{and } a_j \text{ integer.} \tag{2}$$

Let n be the number of feasible patterns and consider the $m \times n$ matrix A with columns a_j, $j = 1, \dots, n$. Note that n can be a very large number. In several practical problems, n can easily exceed 10 million!

The goal of the company is to minimize the number of large rolls used while satisfying customer demand. Let x_j be the number of large rolls cut according to pattern j. Then, the problem under consideration is

$$\begin{array}{lrcl} \min & \sum_{j=1}^{n} x_j & & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j & = & b_i, \ i = 1, \dots, m \\ & x_j & \geq & 0, \ j = 1, \dots, n. \end{array} \tag{3}$$

Technically, each x_j should be an integer and the cutting stock problem is actually an integer program, which is the linear program (3) together with the additional constraints that each x_j should be an integer. However, one can round an optimal solution to (3) to a feasible solution of the integer program, which is also fairly close to the optimal solution of this integer program.

Solving the linear program (3) is hard, since the number of columns in the matrix A can be very large. However, it is possible to modify the revised simplex method to tackle this problem as follows:

1. To find an initial basic feasible solution, we choose the jth pattern to consist of $\lfloor \frac{r}{w_j} \rfloor$ rolls of width w_j and none of the other widths. This gives $A_B = \begin{pmatrix} \lfloor \frac{r}{w_1} \rfloor & & \\ & \ddots & \\ & & \lfloor \frac{r}{w_m} \rfloor \end{pmatrix}$ where $\lfloor x \rfloor$ denotes the *floor* of x. The initial basic feasible solution is given by $x_B = A_B^{-1} b$.

2. In the 2nd step we compute $y = (A_B^T)^{-1} c_B$ as usual. We then need the vector s_N, i.e., $s_i = c_i - a_i^T y$, $i \in N$ (N is the indices of the non-basic variables) to determine the column of A that enters the basis. Since we do not have all the nonbasic columns of A at hand, we consider the problem of minimizing s_i over all $i = 1, \dots, n$ to determine the entering column. Note that $c_i = 1$, $i = 1, \dots, n$. This is the same as maximizing $y^T a_i$ over all $j = 1, \dots, n$. If the maximum value is less than or equal to 1, then $s_i \geq 0$, $i = 1, \dots, n$ and hence for all $i \in N$, and we are optimal! (Remember our LP is a minimization problem and so the optimality condition is $s_N \geq 0$ instead!). On the other hand, if the maximum value is greater than 1, the column a_j corresponding to

a maximizing j enters the basis. We are now left with the task of finding a pattern $j = 1, \ldots, n$ that maximizes $y^T a_j$. Using, the conditions (2) for a feasible pattern, this problem can be formulated as the knapsack problem

$$\begin{array}{lrcl} \max & y^T a & & \\ \text{s.t.} & \sum_{i=1}^{m} w_i a_i & \leq & r \\ & a_i & \geq & 0, \ \ i = 1, \ldots, m, \\ & a_i & \text{is} & \text{integer}, \ \ i = 1, \ldots, m. \end{array} \tag{4}$$

The knapsack problem can be solved using a branch-and-bound algorithm (see Box 13.1 on page 206 of Chvátal). Once a solution a is obtained, we test its objective value against 1. If $y^T a > 1$, the solution a enters as a column in the basis matrix. Else, the current basic feasible solution is optimal!

3. The remaining steps of the revised simplex method are unchanged.

3 Details of the project

1. You will work in a group consisting of 3 students. The entire project is due in class on Tuesday, the 5th of December.

2. Read Chapter 13 in Chvátal carefully before beginning the project. Other references for the project include the original papers by Gilmore & Gomory [1, 2] on this problem, and the survey paper by Lübbecke & Desrosiers [3] on column generation. These papers can be downloaded from the course webpage.

3. Each group will submit a 4-5 page report that describes the cutting stock problem, the details of the algorithm as applied to the cutting stock problem, and your computational experiences with the algorithm on the two sample problems (see part (6) below!).

4. You will write a computer program that implements the column generation algorithm for solving the cutting stock problem in MATLAB. The algorithm will take m (the number of consumer widths), w (the vector of consumer widths of size m), b (the vector of consumer demands for these widths, also of size m), and r (width of the company's supply rolls) as input. It will return an optimal basic feasible solution x_B, the objective value of the optimal solution, and the optimal basis matrix A_B as output. The algorithm will call the branch-and-bound algorithm for the knapsack problem (see Box 13.1 on page 206 of Chvátal) as a subroutine. Try to implement your own subroutine (preferred!) Else, you can download Kartik's branch and bound algorithm for solving mixed integer programs from the course webpage and use it in your subroutine (the branch and bound algorithm in Box 13.1 will be MUCH faster than Kartik's general algorithm for mixed integer programs!)

5. Explain how you will round the optimal solution to (3) obtained in part (4) into a feasible solution for the integer program.

6. Test your algorithm in part (4) and the rounding procedure in part (5) on Problem 13.2 parts a) and b) on page 211 of Chvátal. What are the solutions to these two cutting stock problems? You should include the solutions to these two problems along with a brief summary of your findings in the report.

7. Chvátal describes a procedure to find a good initial basic feasible solution for the cutting stock problem on page 207 of his book. Implement this procedure in your algorithm from part (4). Is there a reduction in the number of simplex iterations? Include a brief summary of your findings in your report.

References

[1] P.C. Gimore and R.E. Gomory, *A linear programming approach to the cutting stock problem*, Operations Research, 9(1961), pp. 849-859.

[2] P.C. Gilmore and R.E. Gomory, *A linear programming approach to the cutting stock problem, Part II*, Operations Research, 11(1963), pp. 863-888.

[3] M.E. Lübbecke and J. Desrosiers, *Selected topics in column generation*, Operations Research, 53(2005), pp. 1007-1023.